STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 3, September 2001

A FORMULA FOR THE MEAN CURVATURE OF AN IMPLICIT REGULAR SURFACE

CORNEL PINTEA

Abstract. In this paper we will find a formula for the absolute value of the mean curvature of an implicit regular surface $(S)\, f(x,y,z)=a$, expressed in terms of the partial derivatives of the function f.

1. **Introduction**

The most used formulas for the Gaussian curvature or for the mean curvature of a regular surface are those that are expressed locally in terms of the coefficients of the first and second fundamental forms.

However for an implicit regular surface (S) $f(x,y,z)=a$ there exists a formula for the Gaussian curvature expressed in terms of the partial derivatives of the function f, that is,

$$K=-\frac{1}{||\overrightarrow{\nabla} f||^4}\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} & f_x \\ f_{yx} & f_{yy} & f_{yz} & f_y \\ f_{zx} & f_{zy} & f_{zz} & f_z \\ f_x & f_y & f_z & 0 \end{vmatrix} \tag{1}$$

In this paper we are going to prove a similar formula for the absolute value of the mean curvature of an implicit regular surface.

For the mean curvature H of a regular surface S we have the following local formula

$$H=\frac{1}{2}\cdot\frac{eG-2fF+gE}{EG-F^2} \tag{2}$$

where

$$E=\overrightarrow{r}_u\cdot\overrightarrow{r}_u=||\overrightarrow{r}_u||^2,\ F=\overrightarrow{r}_u\cdot\overrightarrow{r}_v,\ G=\overrightarrow{r}_v\cdot\overrightarrow{r}_v=||\overrightarrow{r}_v||^2$$

are the coefficients of the first fundamental form and

$$e=\frac{(\overrightarrow{r}_u,\overrightarrow{r}_v,\overrightarrow{r}_{uu})}{\sqrt{EG-F^2}},\ f=\frac{(\overrightarrow{r}_u,\overrightarrow{r}_v,\overrightarrow{r}_{uv})}{\sqrt{EG-F^2}},\ g=\frac{(\overrightarrow{r}_u,\overrightarrow{r}_v,\overrightarrow{r}_{vv})}{\sqrt{EG-F^2}}$$

are the coefficients of the second fundamental form with respect to the local parametrization $r : U \to S$, compatible with the orientation of the surface.

Let $V \subseteq \mathbf{R}^3$ be an open set, $f : V \to \mathbf{R}$ be a differentiable function and $a \in Im\, f$ be a regular value of f. It is well known that $S = f^{-1}(a)$ is an orientable regular surface. For $p \in S$, then one of the partial derivatives $f_x(p), f_y(p), f_z(p)$ is non zero, at least. If $f_z(p) \neq 0$, for instance, then, according to the implicit function theorem, the last variable z can be unically expressed by means of the first two variable x and y. In other words the regular surface $S = f^{-1}(a)$ is locally, around the point p, the graph of a function $z = z(x,y),\ (x,y) \in U$, where U is a conveniently chosen open set. Therefore the mapping $r : U \to S,\ r(x,y) = (x,y,z(x,y))$ is a local parametrization of S at p, namely $f(x,y,z(x,y)) = a,\ \forall (x,y) \in U$. This is the type of local parametrization that we are going to use for all over this paper.

It is very easy to see that $\overrightarrow{r}_x \times \overrightarrow{r}_y = \frac{1}{f_z}\overrightarrow{\nabla} f$ which means that the local parametrization $r : U \to S,\ r(x,y) = (x,y,z(x,y))$ of S at p is compatible with the orientation $\frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}$ of S iff $f_z(p) > 0$ and of course uncompatible iff $f_z(p) < 0$.

In any case the relation

$$2|H| = |\frac{eG - 2fF + gE}{EG - F^2}| \tag{3}$$

holds.

2. **The main formula**

In this section we will prove the already anounced formula for the absolute value of the mean curvature of an implicit regular surface.

Theorem 2.1. *Let $V \subseteq \mathbf{R}^3$ be an open set, $f : V \to \mathbf{R}$ be a smooth function and $a \in Im\, f$ be a regular value of the f. For the absolute value of the mean curvature H of the implicit regular surface $(S)\ f(x,y,z) = a$, at the point $p \in S$, we have the following formula*

$$|H| = \frac{1}{2||\overrightarrow{\nabla} f||}|\Big[\Delta f - (Hess\, f)\Big(\frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}, \frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}\Big)\Big]|, \tag{4}$$

where $\overrightarrow{\nabla} f$ is the gradient of f, Δ is the Laplace's operator and $Hess\, f$ is the Hessian of f, all of them being considered at the point p.

Proof. Assuming that for $p \in f^{-1}(a)$ we have $f_z(p) \neq 0$, it follows that S is locally, around the point p, the graph of a function $z = z(x,y)$, $(x,y) \in U$ and consider the above stated local parametrization $r : U \to S$, $r(x,y) = (x,y,z(x,y))$.

The coefficients of the two fundamental forms are

$$E = 1 + z_x^2, \qquad F = z_x \cdot z_y, \qquad G = 1 + z_y^2$$
$$e = \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}}, \quad f = \frac{z_{xy}}{\sqrt{1+z_x^2+z_y^2}}, \quad g = \frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}}$$

$$2|H| = |\frac{eG - 2fF + gE}{EG - F^2}| = |\frac{(1+(f_x)^2)f_{yy} - 2f_x f_y f_{xy} + (1+f_y)^2 f_{xx}}{[1 + z_x^2 + z_y^2]^{3/2}}|. \tag{5}$$

Because $f(x,y,z(x,y)) = a$, $\forall\,(x,y) \in U$, it follows that

$$\begin{cases} f_x + z_x f_z = 0 \\ f_y + z_y f_z = 0 \end{cases} \text{that is} \begin{cases} z_x = -\frac{f_x}{f_z} \\ z_y = -\frac{f_y}{f_z}. \end{cases} \tag{6}$$

¿From relations (6) we get

$$\begin{cases} z_{xx} = -\frac{\partial}{\partial x}\left[\frac{f_x(x,y,z(x,y))}{f_z(x,y,z(x,y))}\right] = -\frac{f_z^2 f_{xx} - 2f_x f_z f_{xz} + f_x^2 f_{zz}}{f_z^3} \\ z_{xy} = -\frac{\partial}{\partial y}\left[\frac{f_x(x,y,z(x,y))}{f_z(x,y,z(x,y))}\right] = -\frac{f_z^2 f_{xy} - f_y f_z f_{xz} - f_x f_z f_{yz} + f_x f_y f_{zz}}{f_z^3} \\ z_{yy} = -\frac{\partial}{\partial x}\left[\frac{f_y(x,y,z(x,y))}{f_z(x,y,z(x,y))}\right] = -\frac{f_z^2 f_{yy} - 2f_y f_z f_{yz} + f_y^2 f_{zz}}{f_z^3}. \end{cases} \tag{7}$$

Replacing the partial derivatives $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ given by the relations (6), (7)in the formula (5) we obtain

$$2|H| =$$

$$|\frac{\frac{(f_y^2+f_z^2)(f_{xx}f_z^2-2f_x f_z f_{xz}+f_{zz}f_x^2)}{f_z^5} - 2\frac{f_x f_y (f_x f_y f_{zz} - f_y f_z f_{xz} - f_x f_z f_{yz} + f_{xy} f_z^2)}{f_z^5} + \frac{(f_x^2+f_z^2)(f_{yy}f_z^2-2f_y f_z f_{yz}+f_{zz}f_y^2)}{f_z^5}}{\left[\frac{f_x^2+f_y^2+f_z^2}{f_z^2}\right]^{3/2}}|$$

$$= |\frac{|f_z|^3}{f_z^5}\Big[\frac{f_y^2 f_z^2 f_{xx} - 2f_x f_y^2 f_z f_{xz} + f_x^2 f_y^2 f_{zz} + f_z^4 f_{xx} f_x f_z^3 f_{xz} + f_x^2 f_z^2 f_{zz} - 2f_x^2 f_y^2 f_{zz} + 2f_x f_y^2 f_z f_{xz}}{\sqrt{f_x^2 + f_y^2 + f_z^2}^3} +$$

$$+ \frac{2f_x^2 f_y f_z f_{yz} - 2f_x f_y f_z^2 f_{xy} + f_x^2 f_z^2 f_{yy} - 2f_x^2 f_y f_z f_{yz} + f_x^2 f_y^2 f_{zz} + f_z^4 f_{yy} - 2f_y f_z^3 f_{yz} + f_y^2 f_z^2 f_{zz}}{\sqrt{f_x^2 + f_y^2 + f_z^2}^3}\Big]| =$$

$$= |\frac{|f_z|^3}{f_z^5}\,\frac{f_z^2(f_y^2 f_{xx} + f_z^2 f_{xx} - 2f_x f_z f_{xz} + f_x^2 f_{zz} - 2f_x f_y f_{xy} + f_x^2 f_{yy} + f_z^2 f_{yy} - 2f_y f_z f_{yz} + f_y^2 f_{zz})}{||\overrightarrow{\nabla} f||^3}| =$$

$$= |\frac{f_x^2(f_{yy} + f_{zz}) + f_y^2(f_{xx} + f_{zz}) + f_z^2(f_{xx} + f_{yy}) - (Hess\,f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) + f_x^2 f_{xx} + f_y^2 f_{yy} + f_z^2 f_{zz}}{||\overrightarrow{\nabla} f||^3}|$$

where

$$(Hesss\,f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f) = (f_x, f_y, f_z)\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} =$$

$$= f_{xx}f_x^2 + f_{yy}f_y^2 + f_{zz}f_z^2 + 2f_{xy}f_xf_y + 2f_{xz}f_xf_z + 2f_{yz}f_yf_z.$$

Therefore for the absolute value of the mean curvature wee have

$$|H| = \frac{1}{2}\Big|\frac{f_x^2(f_{xx}+f_{yy}+f_{zz}) + f_y^2(f_{xx}+f_{yy}+f_{zz} + f_z^2(f_{xx}+f_{yy}+f_{zz}) - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)}{||\overrightarrow{\nabla} f||^3}\Big| =$$

$$= \frac{1}{2}\Big|\frac{(f_x^2+f_y^2+f_z^2)(f_{xx}+f_{yy}+f_{zz}) - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)}{||\overrightarrow{\nabla} f||^3}\Big| =$$

$$= \frac{1}{2}\Big|\frac{||\overrightarrow{\nabla} f||^2 \cdot \Delta f - (Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)}{||\overrightarrow{\nabla} f||^3}\Big| =$$

$$= \frac{1}{2||\overrightarrow{\nabla} f||}\Big|\Big[\Delta f - (Hess\, f)\Big(\frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}, \frac{\overrightarrow{\nabla} f}{||\overrightarrow{\nabla} f||}\Big)\Big]\Big|.\square$$

Corollary 2.2. *If $V \subseteq \mathbf{R}^3$ is an open set, $f : V \to \mathbf{R}$ is a smooth harmonic mapping and $a \in Im\, f$ is a regular value of f, then for the absolute value of the mean curvature of the implicit regular surface $(S)\ f(x, y, z) = a$ we have the following formula:*

$$|H| = \frac{1}{2||\overrightarrow{\nabla} f||^3}\big|(Hess\, f)(\overrightarrow{\nabla} f, \overrightarrow{\nabla} f)\big|. \tag{8}$$

3. Example

It is well know that the locus of the orthogonal projections of the center of the ellipsoid $(E)\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ on its tangent planes is the so called *pedal surface* of E, that is the regular surface

$$S = \{(x, y, z) \in \mathbf{R}^3 \,|\, (x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2\}\backslash\{0\}.$$

We will compute the absolute value of the mean curvature of the pedal surface of E in its points.

For this purpose consider $p = (x_0, y_0, z_0) \in S$, the function

$$f : \mathbf{R}^3\backslash\{0\} \to \mathbf{R},\ f(x, y, z) = (x^2 + y^2 + z^2)^2 - a^2x^2 - b^2y^2 - c^2z^2$$

and observe that $S = f^{-1}(0)$.

The partial derivatives of first and second order of f are

$$\begin{aligned} f_x &= 4x(x^2+y^2+z^2)-2a^2x \\ f_y &= 4y(x^2+y^2+z^2)-2b^2y \\ f_z &= 4z(x^2+y^2+z^2)-2c^2z \end{aligned}$$

$$\begin{aligned} f_{xx} &= 4(x^2+y^2+z^2)+8x^2-2a^2 \quad f_{xy}=f_{yx}=8xy \quad f_{xz}=f_{zx}=8xz \\ f_{yy} &= 4(x^2+y^2+z^2)+8y^2-2b^2 \quad f_{yz}=f_{zy}=8yz \\ f_{zz} &= 4(x^2+y^2+z^2)+8z^2-2c^2 \end{aligned}.$$

Therefore in the points (x,y,z) of the regular surface S we have $||\overrightarrow{\nabla} f||^2 = 4(a^4x^2+b^4y^2+c^4z^2)$, or equivalent $||\overrightarrow{\nabla} f|| = 2(a^4x^2+b^4y^2+c^4z^2)^{1/2}$. Observe that $||\overrightarrow{\nabla} f|| \neq 0$ in all the points of the surface $S=f^{-1}(0)$. Therefore the critical set of f doesn't intersects the level set $S=f^{-1}(0)$, this being of course an argument on the regularity of S.

On the other hand $\Delta f = 20(x^2+y^2+z^2)-2(a^2+b^2+c^2)$ and

$$(Hessf)(\overrightarrow{\nabla} f,\overrightarrow{\nabla} f) = f_{xx}f_x^2+f_{yy}f_y^2+f_{zz}f_z^2+2f_{xy}f_xf_y+2f_{xz}f_xf_z+2f_{yz}f_yf_z =$$

$$= (4(x^2+y^2+z^2)+8x^2-2a^2)[16x^2(x^2+y^2+z^2)^2-16a^2x^2(x^2+y^2+z^2)+4a^4x^2]+$$

$$+(4(x^2+y^2+z^2)+8y^2-2b^2)[16y^2(x^2+y^2+z^2)^2-16b^2y^2(x^2+y^2+z^2)+4b^4y^2]+$$

$$+(4(x^2+y^2+z^2)+8z^2-2c^2)[16z^2(x^2+y^2+z^2)^2-16c^2z^2(x^2+y^2+z^2)+4c^4z^2]+$$

$$+16xy[4x(x^2+y^2+z^2)-2a^2x][4y(x^2+y^2+z^2)-2a^2y]+$$

$$+16xz[4x(x^2+y^2+z^2)-2a^2x][4z(x^2+y^2+z^2)-2c^2z]+$$

$$+16yz[4y(x^2+y^2+z^2)-2a^2y][4z(x^2+y^2+z^2)-2c^2z] =$$

$$= 48(x^2+y^2+z^2)(a^4x^2+b^4y^2+c^4z^2).$$

Replacing all of these values considered in p, in the formula (4), we obtain

$$|H_{S_1}(p)| =$$

$$= \frac{1}{4(a^4x_0^2+b^4y_0^2+c^4z_0^2)^{1/2}}\left|20(x_0^2+y_0^2+z_0^2)-2(a^2+b^2+c^2)-\frac{48(x_0^2+y_0^2+z_0^2)(a^4x_0^2+b^4y_0^2+c^4z_0^2)}{4(a^4x_0^2+b^4y_0^2+c^4z_0^2)}\right| =$$

$$= \frac{|4(x_0^2+y_0^2+z_0^2)-(a^2+b^2+c^2)|}{2(a^4x_0^2+b^4y_0^2+c^4z_0^2)^{1/2}} = \left|2\sqrt{\frac{a^2x_0^2+b^2y_0^2+c^2z_0^2}{a^4x_0^2+b^4y_0^2+c^4z_0^2}}-\frac{1}{2}\frac{a^2+b^2+c^2}{\sqrt{a^4x_0^2+b^4y_0^2+c^4z_0^2}}\right|.$$

References

[Ca] Carmo, M., do, *Differentiable Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey, 1976.

[Fe] Fédenko, A., *Recueil D'exercises de Géométrie Différentielle* Éditions MIR · Moscou, 1982.

[Gr] Gray, A., *Modern Differential Geometry of Curves and Surfaces*, CRC Press, 1993.

[Pi] Pintea, C., *Geometrie. Elemente de Geometrie Analitică. Elemente de Geometrie Diferenţială a Curbelor şi Suprafeţelor*, Presa Universitară Clujeană, 2001.

"Babeş-Bolyai" University, Faculty of Mathematics,
Str. M. Kogălniceanu 1, 3400 Cluj-Napoca, Romania
E-mail address: cpintea@math.ubbcluj.ro