STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 3, September 2001

A DIFFUSION PROBLEM IN A CIRCULAR DOMAIN IN A POROUS LAYER

T. GROŞAN AND I. POP

1. Introduction

Transport and flow phenomena in porous media or industrial synthetic porous materials, arise in many diverse fields of science and engineering, ranging from agricultural, biomedical, construction, ceramic, chemical, and petroleum engineering to food and soil science, and powder technology. Fifty percent of more of the original oil-in-plan is left in a typical oil reservoir by traditional recovery techniques. Oil recovery processes constitute only a small fraction of an enormous, and still rapidly growing, literature on porous media. In addition to oil recovery processes, the closely related areas of soil science and hydrology are perhaps the best – established topics. The study of groundwater flow and the restoration of aquifers that have been contaminated by various pollutants are important current areas of research in porous media problems. The construction industry, transmission of water by building materials is also an important problem that uses porous media. Phenomena involving porous media are also numerous. Recent books by Ingham and Pop [1], Nield and Bejan [2], Vafai [3] and Pop and Ingham [4] on transport phenomena in porous media clearly demonstrate that flows in porous media are becoming a classical subject, once where earlier developments have been confirmed by a large number of studies.

The present paper studies a diffusion problem in a porous layer of circular form and thickness Δz. We suppose that the pressure p does not vary with height. Thus, the fluid motion is reduced to two – dimensional flow in a circular domain. We assume that the domain's boundary is impermeable, that at the moment $t = 0$ the fluid has an initial pressure, p_i, and that a negative source is placed in the centre of the domain. We will study the evolution of the pressure on time.

2. **Basic Equation**

We consider the two – dimensional flow of a viscous and compressible fluid generated by a negative source of debit q placed in the porous layer. We study the fluid motion in a circular domain where the source is placed in the centre of the domain. The problem is described by the continuity equation, Darcy's law and the state equation as established by Cretu [5] or Ungureanu et al. [6]:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) - M_s(x, y, t) = -\frac{\partial}{\partial t}(m\rho) \tag{1}$$

$$u = -\frac{K}{\mu}\frac{\partial p}{\partial x}, v = -\frac{K}{\mu}\frac{\partial p}{\partial y} \tag{2}$$

$$\rho = \rho_0 e^{\beta(p-p_0)} \tag{3}$$

where x and y are Cartesian coordinates, u and v are velocity components along x and y axes, respectively, K is the permeability of porous medium, ρ is the density, μ is the viscosity, p is the pressure, ρ_0 is te density at the atmospheric pressure p_0 and β is the compressibility coefficient defined as

$$\beta = \frac{1}{\rho}\frac{d\rho}{dp} \tag{4}$$

Because the compressibility coefficient, β , is small equation (3) can be expressed as:

$$\rho \approx \rho_0 [1 + \beta(p - p_0)] \tag{5}$$

If we assume that the porous medium is homogeneous (K is constant in x and y directions, respectively), μ is independent of the pressure p and that M_s = q (constant), Eqs.(1) – (5) reduces, after some algebra to the following equation:

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial p}{\partial y}\right) - q = c\frac{\partial p}{\partial t} \tag{6}$$

and it describes the flow of a viscous fluid trough porous medium. In this equation K_x and K_y denotes the permeability in x and y direction, respectively, q is the debit and c is the hydraulic capacity.

3. **Application**

Because the flow domain is circular the flow is symmetric. Thus, Eq. (5) is written in polar coordinates (r,θ) as follows (see, for example, Kohr [7]):

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\mu}{k}q = \frac{m\beta\mu}{k}\frac{\partial p}{\partial t} \tag{7}$$

where $\partial/\partial\theta = 0$ has been used. Equation (7) is now written in the non – dimensional form by using the new variables

$$r^* = \frac{r}{R}, p^* = \frac{p}{p_i}, t^* = \frac{kt}{R^2 m\beta\mu}, q^* = \frac{R^2\mu q}{kp_i} \tag{8}$$

where R is the radius of the circular domain. Substituting the variables (8) into Eq. (7), it becomes

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + q = \frac{\partial p}{\partial t} \tag{9}$$

where the star has been dropped.

We shall assume now that the boundary of the circular domain is impermeable and that at t = 0 the initial pressure, p_i, is constant and equal with one. Thus, the initial and boundary conditions of Eq. (8) are

$$p(r,0) = 1, \frac{\partial p}{\partial r}(1,t) = 0 \tag{10}$$

Further, we notice that at r = 0, we have

$$\lim_{r\to 0}\frac{\frac{\partial p}{\partial r}}{r} = \frac{\partial^2 p}{\partial r^2} \tag{11}$$

Therefore, Eq. (9) can be written as

$$r = 0: \quad 2\frac{\partial^2 p}{\partial r^2} + q = \frac{\partial p}{\partial t} \tag{12}$$

$$r \neq 0: \quad \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + q = \frac{\partial p}{\partial t} \tag{13}$$

We will use the finite difference operators for the derivatives, which appear in Eqs. (12) and (13) (see Ixaru [8]):

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{1}{2h}\left(p_{i+1}-p_{i-1}\right) \\ \frac{\partial^2 p}{\partial r^2} &= \frac{1}{h^2}\left(p_{i-1}-2p_i+p_{i+1}\right) \\ \frac{\partial p}{\partial r} &= \frac{1}{\Delta t}\left(p_{i,n+1}-p_{i,n}\right) \end{aligned} \tag{14}$$

For r = 0 (i = 0) we have

$$\frac{\partial p}{\partial r} = \frac{1}{2h}\left(p_1-p_{-1}\right)=0 \tag{15}$$

and we find that $p_1 = p_{-1}$ and the second order derivative becomes

$$\frac{\partial^2 p}{\partial r^2} = \frac{1}{h^2}\left(p_{-1}-2p_0+p_1\right)=\frac{2}{h^2}\left(p_1-p_0\right) \tag{16}$$

For r = 1 (i = n) we obtain from the condition of boundary impermeability

$$\frac{\partial p}{\partial r} = \frac{1}{2h}\left(p_{n+1}-p_{n-1}\right)=0 \tag{17}$$

so that $p_{n+1} = p_{n-1}$, and the second order derivative becomes

$$\frac{\partial^2 p}{\partial r^2} = \frac{1}{h^2}\left(p_{n-1}-2p_n+p_{n+1}\right)=\frac{2}{h^2}\left(p_{n-1}-p_n\right) \tag{18}$$

The debit function has a nonzero value only in the origin, so that we have:

$$q(r,t)=\begin{cases} \dfrac{R^2\mu q}{kp_i}0 & \text{for} \quad r=0 \\ 0 & \text{for} \quad r\neq 0 \end{cases} \tag{19}$$

Using (14) – (19) the equations (12) – (13) become:

$$\begin{aligned} j &= 0: p_{0,n+1}=\frac{2\Delta t}{h^2}\left(p_1-p_0\right)+q\Delta t+p_{0,n} \\ j &> 0: p_{j,n+1}=\frac{\Delta t}{h^2}\left(p_{j-1}-2p_j+p_{j+1}\right)+\frac{\Delta t}{jh}\left(p_{j+1}-p_{j-1}\right)+p_{j,n} \\ j &= n: p_{n,n+1}=\frac{2\Delta t}{h^2}\left(p_{n-1}-p_n\right)+p_{n,n} \end{aligned} \tag{20}$$

Equations (20) form an explicit scheme of finite difference for our problem. For the study of convergence we have used a Fourier analysis. We write the pressure p like a Fourier series (see Morton and Mayers [9]):

$$p_{j,n} = \lambda^n e^{ik(jh)} \tag{21}$$

where λ is the amplification parameter and i $=\sqrt{-1}$. After some algebra using (20) and (21) we found:

$$\lambda = 1 - 4\frac{\Delta t}{h^2} \tag{22}$$

Because the condition of converge is that the amplification parameter must be between −1 and 1 we have the condition

$$\frac{\Delta t}{h^2} \leq \frac{1}{2} \tag{23}$$

4. **Results and Discussion**

The Eqs. (20) have been integrated using the time steep $\Delta t = 0.01$ and the spatial steep h = 0.2 and we can see from Eq. (23) that the convergence condition is satisfied. In the Table 1. we have presented the results for q = -1. Obviously the value of the pressure is decreasing in the entire domain, but the effect is more present in the center of the domain. At different moment of time the pressure shape is the same, but the values are lowers, as can see in Figure 1. This behaviour is similar to the one described by Cretu [5] for a rectangular domain:

TABLE 1. The values of the pressure at different moment of time

Δt	h = 0	h = 0.2	h = 0.4	h = 0.6	h = 0.8	h = 1
0	1	1	1	1	1	1
0.5	0.93955	0.95839	0.96902	0.97565	0.97930	0.98042
1	0.91090	0.92975	0.94042	0.94707	0.95074	0.95188
2	0.85372	0.87257	0.88324	0.88989	0.89355	0.89470
3	0.79654	0.81540	0.82606	0.83271	0.83628	0.83752
4	0.73936	0.75822	0.76888	0.77553	0.77920	0.78034
5	0.68218	0.70104	0.71170	0.71835	0.72202	0.72316
6	0.62500	0.64386	0.65452	0.66117	0.66484	0.66598
7	0.56782	0.58668	0.59734	0.60399	0.60766	0.60880
8	0.51064	0.52950	0.54016	0.54681	0.55048	0.55162
9	0.45346	0.47232	0.48298	0.48963	0.49330	0.49444
10	0.39628	0.41514	0.42580	0.43245	0.43612	0.43726

Fig.1. The variation of pressure p at different moment of time

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Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania