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FIRST ORDER DIFFERENTIAL SUBORDINATIONS AND INEQUALITIES IN A BANACH SPACE

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Abstract. Let E be a complex Banach space and let $B = \{x \in E : \|x\| < 1\}$ be the unit ball in E. Let $p : B \to \mathbb{C}$ be holomorphic in B and let q be holomorphic and univalent in the unit disc U. We prove that if p satisfies some differential subordinations and inequalities, then $p(B) \subset q(U)$. Applications of these results are presented.

1. **Introduction**

S. Gong and S.S. Miller [1] have dealt with holomorphic functions defined on a complete circular domain in $\mathbb{C}^n$, which satisfy certain partial differential inequalities or subordinations. In this paper we consider similar relationships for holomorphic functions from the unit ball B into $\mathbb{C}$.

The following sets $\{x \in E : \ \|x\| < r \leq 1\}$ and $\{x \in E : \ \|x\| \leq r \leq 1\}$ will be denoted B_r, respectively $\overline{B}_r$.

Let $H(B_r)$, $r \in (0, 1]$ be the class of functions $f : B_r \to \mathbb{C}$ that are holomorphic in B_r, i.e. have the Fréchet derivative $f'(x)$ in each point $x \in B_r$.

2. **First order differential subordinations**

Lemma 1. *Let $r_0 \in (0, 1)$ and let $f \in H(\overline{B}_{r_0})$ with $f(0) = 0$ and $f(x) \not\equiv 0$. If $x_0 \in \overline{B}_{r_0}$ and*

$$|f(x_0)| = \max\{|f(x)| : \ x \in \overline{B}_{r_0}\} \tag{1}$$

then there exists $m \in \mathbb{C}$ with Re $m \geq 1$ *such that*

$$f'(x_0)(x_0) = m f(x_0). \tag{2}$$

Proof. We have $zx \in B_{r_0}$ for all $z \in U$ and $x \in \overline{B}_{r_0}$. We consider the function $g(z) = \dfrac{f(zx_0)}{f(x_0)}$, for $z \in U$. From (1) we obtain

$$|g(z)| = \left|\frac{f(zx_0)}{f(x_0)}\right| < 1, \quad \text{for all} \quad z \in U.$$

Since $g(0) = 0$, we can apply Schwarz's lemma to obtain $|g(z)| \leq |z|$, $z \in U$ and thus

$$\left|\frac{f(zx_0)}{f(x_0)}\right| \leq |z|, \quad \text{for} \quad z \in U.$$

At the point $z = r$, $r \in (0,1)$ we have

$$\operatorname{Re} \frac{f(rx_0)}{f(x_0)} \leq r. \tag{3}$$

A simple calculation leads to

$$\frac{f'(x_0)(x_0)}{f(x_0)} = \frac{d}{dr}\left[\frac{f(rx_0)}{f(x_0)}\right]\Bigg|_{r=1} = \lim_{r\nearrow 1} \frac{f(rx_0) - f(x_0)}{(r-1)f(x_0)} = \lim_{r\nearrow 1}\left[1 - \frac{f(rx_0)}{f(x_0)}\right]\frac{1}{1-r}.$$

Taking real parts and using (3) we obtain

$$\operatorname{Re} \frac{f'(x_0)(x_0)}{f(x_0)} \geq \lim_{r\nearrow 1}(1-r)\frac{1}{1-r} = 1,$$

which proves the lemma.

We will extend the ideas in Lemma 1, but first we need to consider the following class of functions.

Definition 1. We denote by Q the set of functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \{\zeta \in \partial U : \lim_{z\to\zeta} q(z) = \infty\}$ and are such that $q'(\zeta) \neq 0$, for $\zeta \in \partial U \setminus E(q)$.

Lemma 2. *Let $q \in Q$ and let $p \in H(B)$ with $p(0) = q(0)$. If $p(B) \not\subset q(U)$ then there exist $r_0 \in (0,1)$, $x_0 \in \overline{B}_{r_0}$ and $\zeta_0 \in \partial U \setminus E(q)$ such that*

(i) $p(x_0) = q(\zeta_0)$

(ii) $p'(x_0)(x_0) = m\zeta_0 q'(\zeta_0)$, where $\operatorname{Re} m \geq 1$.

Proof. Since $p(0) = q(0)$ and $p(B) \not\subset q(U)$ there exists $r_0 \in (0,1)$ such that $p(B_{r_0}) \subset q(U)$ and $p(\overline{B}_{r_0}) \cap q(\partial U) \setminus E(q) \neq \emptyset$. Hence there exist $x_0 \in \overline{B}_{r_0}$ and $\zeta_0 \in \partial U \setminus E(q)$ such that $p(x_0) = q(\zeta_0)$. If we let $f(x) = q^{-1}(p(x))$, for $x \in \overline{B}_{r_0}$, then f is holomorphic in $\overline{B}_{r_0}$ and satisfies $|f(x_0)| = |\zeta_0| = 1$, $f(0) = 0$ and $|f(x)| \leq 1$, for $x \in \overline{B}_{r_0}$. Thus f satisfies the conditions of Lemma 1 and we obtain that there eixsts $m \in \mathbb{C}$, with $\operatorname{Re} m \geq 1$ such that $f'(x_0)(x_0) = mf(x_0)$. Since

$p(x) = q(f(x))$, we have $p'(x) = q'(f(x))f'(x)$ and using $\zeta_0 = f(x_0)$, we obtain $p'(x_0)(x_0) = q'(f(x_0))f'(x_0)(x_0) = m\zeta_0 q'(\zeta_0)$.

Definition 2. Let Ω be a set in $\mathbb{C}$, $q \in Q$. We define $\psi[\Omega, q]$ to be the class of functions $\psi : \mathbb{C}^2 \times B \to \mathbb{C}$ that satisfy the condition:

$$\psi(r, s; x) \notin \Omega, \quad \text{whenever} \quad r = q(\zeta), \quad s = m\zeta q'(\zeta),$$

$$x \in B, \quad \zeta \in \partial U \setminus E(q) \quad \text{and} \quad \text{Re}\, m \geq 1.$$

We are now prepared to present the main result of this section.

Theorem 1. *Let $\psi \in \psi[\Omega, q]$. If $p \in H(B)$ with $p(0) = q(0)$ and if p satisfies*

$$\psi(p(x), p'(x)(x); x) \in \Omega, \quad \textit{for} \quad x \in B \tag{4}$$

then $p(B) \subset q(U)$.

Proof. Assume $p(B) \not\subset q(U)$. By Lemma 2 there exist $x_0 \in B$, $\zeta_0 \in \partial U \setminus E(q)$ and $m \in \mathbb{C}$ with Re $m \geq 1$ that satisfy (i), (ii) of Lemma 2. Using these conditions with $r = p(x_0)$, $s = p'(x_0)(x_0)$ and $x = x_0$ in Definition 2 we obtain

$$\psi(p(x_0), p'(x_0)(x_0); x_0) \notin \Omega.$$

Since this contradicts (4) we must have $p(B) \subset q(U)$.

We next apply Theorem 1 to two important particular cases corresponding to $q(U)$ being the unit disc and $q(U)$ being the right half-plane.

If we take $q(z) = z$ in Definition 2 and Theorem 1 we obtain the following result.

Corollary 1. *Let Ω be a set in $\mathbb{C}$ and let $\psi : \mathbb{C}^2 \times B \to \mathbb{C}$ be such that*

$$\psi(e^{i\theta}; me^{i\theta}; x) \notin \Omega, \quad \textit{whenever} \quad x \in B, \quad \theta \in \mathbb{R} \quad \textit{and} \quad \text{Re}\, m \geq 1. \tag{5}$$

If $p \in H(B)$ with $p(0) = 0$ and if p satisfies

$$\psi(p(x), p'(x)(x); x) \in \Omega, \quad \textit{for} \quad x \in B$$

then $|p(x)| < 1$, for $x \in B$.

If we take $q(z) = \dfrac{1+z}{1-z}$ in Definition 2 and Theorem 1 we obtain:

Corollary 2. *Let Ω be a set in $\mathbb{C}$ and let $\psi : \mathbb{C}^2 \times B \to \mathbb{C}$ be such that*

$$\psi(ai, s; x) \notin \Omega, \quad \textit{whenever} \quad x \in B, \quad a \in \mathbb{R}, \quad \textit{and} \quad \text{Re}\, s \leq -\frac{1+a^2}{2}. \tag{6}$$

If $p \in H(B)$ *with* $p(0) = 1$ *and if* p *satisfies*

$$\psi(p(x), p'(x)(x); x) \in \Omega, \quad \textit{for} \quad x \in B$$

then Re $p(x) > 0$*, for* $x \in B$.

3. **Examples**

In this section we present a seris of examples of differential inequalities by applying the two corollaries of the previous section.

Example 1. Let $\Omega = U$ and let $\psi(r, s; x) = \alpha(|r| + |s|) + \beta\|x\|$, where $\alpha \geq \frac{1}{2}$ and $\beta \geq 0$. If $p \in H(B)$ with $p(0) = 0$, then

$$\alpha(|p(x)| + |p'(x)(x)|) + \beta\|x\| < 1 \Rightarrow |p(x)| < 1.$$

Proof. To use Corollary 1 we need to shoe that the condition (5) is satisfied. This follows since

$$|\psi(e^{i\theta}, me^{i\theta}; x)| = \Big|\alpha(1 + |m|) + \beta\|x\|\Big| \geq \alpha(1 + |m|) \geq \alpha(1 + \text{Re } m) \geq 2\alpha \geq 1.$$

Remark. When $\alpha = \frac{1}{2}$ and $\beta = 0$ we have

$$|p(x)| + |p'(x)(x)| < 2 \Rightarrow |p(x)| < 1.$$

The proof of the following example also follows from Corollary 1.

Example 2. Let $\Omega = U$ and let $\psi(r, s; x) = \alpha(x)r + \beta s$, where $\beta \geq 0$ and $\alpha : B \to \mathbb{C}$ such that Re $\alpha(x) \geq 1 - \beta$. If $p \in H(B)$ with $p(0) = 0$, then

$$|\alpha(x)p(x) + \beta p'(x)(x)| < 1 \Rightarrow |p(x)| < 1.$$

Example 3. Let $\Omega = \{z \in \mathbb{C} : \text{ Re } z > 0\}$ and let $\psi(r, s; x) = r^2 + s$. If $p \in \mathcal{H}(B)$ with $p(0) = 1$, then

$$\text{Re } [p^2(x) + p'(x)(x)] > 0 \Rightarrow \text{Re } p(x) > 0.$$

Proof. To use Corollary 2 we need to show that the condition (6) is satisfied. This follows since

$$\text{Re } \psi(ai, s; x) = -a^2 + \text{Re } s \leq \frac{-3a^2 - 1}{2} < 0.$$

The proof of the following example also follows from Corollary 2.

Example 4. Let $\Omega = \{z \in \mathbb{C} : \text{ Re } z > 0\}$ and let $\psi(r, s; x) = \alpha(x)r + \beta s$, where $\beta \geq 0$ and $\alpha : B \to \mathbb{C}$ such that $|\text{Im } \alpha(x)| \leq \beta$. If $p \in H(B)$ with $p(0) = 1$, then

$$\text{Re } [\alpha(x)p(x) + \beta p'(x)(x)] > 0 \ \Rightarrow \ \text{Re } p(x) > 0.$$

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