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M-LINEAR CONNECTION ON THE SECOND ORDER REONOM BUNDLE

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Abstract. The $T^2M \times R$ bundle represents the total space of a time dependent geometry of second order. In this bundle it is studied a special class of derivation rules, named M-linear connections.. There are given their characterization and it is proved their existence. Finally there are studied geometrical properties of one M-linear connection.

1. **Introduction**

The study of the time dependent Lagrange geometry (geometry of the reonom Lagrange spaces) was imposed from considerations of mechanic ,a systematically study of this is finding in the M.Anastasiei and H.Kawaguchi paper [1],[2],[3].

On the other hand, research from the last years imposed into attention the considerations in the superior order geometries where the total space is the prolongation of k order of the TM tangent bundle of a differential manifold or an associated bundle named the osculator bundle of k order ([5],[8],[13]). From calculation reasons we will approach here the case $k = 2$.

The study of the second order reonom bundle $E = T^2M \times R$ was done by us in a previous work([6],[7]).

Let M be a differentiable manifold, $dimM = n$, $x = (x^i)$ the local coordinates in a map (U, φ). We are considering T^2M the 2-jets bundle to the tangent curves in $x \in M$. Locally on T^2M the coordinates are $u = (x^i, y^i, z^i)$ with the following rule of change on the intersection of two local maps:

$$\widetilde{x}^i = \widetilde{x}^i(x^j) \tag{1.1}$$

$$\widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^i$$

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$$\widetilde{z}^i = \frac{1}{2}\frac{\partial \widetilde{y}^i}{\partial x^k} y^k + \frac{\partial \widetilde{x}^i}{\partial x^k} z^k$$

T^2M has a structure of fibre bundle over R^{2n} space ,which is not vectorial one.

The reonom bundle of second order is the bundle of direct product $E = T^2M \times R$, in which variable on R is denoted by t and it is considered in applications as being the time. In respect to the (1.1) changes on E we will haw also and $\widetilde{t} = t$.

Taking as a base the E manifold, we will develop a geometrical techniques of derivation the sections on TE. The tangent space T_uE present approaching difficulties due to the fact that the natural bases $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial t}\}$ it is changing with the two order derivatives of $\frac{\partial \widetilde{x}^i}{\partial x^j}$.

In order to eliminate this inconvenient we will consider an adapted base of a nonlinear connection on E.

Let $\Pi_2 : E \to M$ the canonical projection and Π_2^* the cotangent map,$\mathcal{V}^2E = Ker\Pi_2^*$ the vertical subbundle of second order. We are considering also the bundle $\Pi_{12} : E \to TM \times R$ and $\mathcal{V}E = Ker\Pi_{12}^*$ the vertical subbundle of first order, that at his turn, is subbundle of the vertical bundle of second order, through his natural structure. Local bases in $\mathcal{V}E$ and $\mathcal{V}^2E$ are respectively $\{\frac{\partial}{\partial x^i}, \frac{\partial}{t}\}$ and $\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial t}\}$.

Definition 1. A *nonlinear connection* on E is a splitting of the TE in the sum $TE = \mathcal{V}^2E \oplus \mathcal{N}E$,where $\mathcal{N}E$ will be named the normal subbundle of E.

Locally, a base in $u \to \mathcal{N}_uE$ distribution is given by $\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \mathcal{N}_i^j \frac{\partial}{\partial y^j} - \mathcal{M}_i^j \frac{\partial}{\partial z^j} - \mathcal{K}_I^0 \frac{\partial}{t}\}$ We are imposing further the conditions of global definition of the adapted fields $\{\frac{\delta}{\delta y^i}\}$ and $\{\frac{\delta}{\delta x^i}\}$,

$$\frac{\delta}{\delta x^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta \widetilde{x}^j} \quad and \quad \frac{\delta}{\delta y^i} = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\delta}{\delta \widetilde{y}^j} \tag{1.2}$$

Consequently, we are obtaining the next changing rules of the nonlinear connection coefficients on E.

$$\widetilde{\mathcal{N}}\,{}^r_k \frac{\partial \widetilde{x}^r}{\partial x^k} = \frac{\partial \widetilde{x}^r}{\partial x^k}\mathcal{N}_i^k - \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^k} z^k + \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^k} y^i - \frac{1}{2}\frac{\partial^3 \widetilde{x}^r}{\partial x^i \partial x^j \partial x^k} y^i y^k .. \tag{1.3}$$

$$\widetilde{\mathcal{M}}\,{}^r_k \frac{\partial \widetilde{x}^k}{\partial x^i} = \frac{\partial \widetilde{x}^r}{\partial x^k}\mathcal{M}_i^k - \frac{\partial^2 \widetilde{x}^r}{\partial x^i \partial x^k} y^k \tag{1.4}$$

$$\widetilde{\mathcal{K}}\,_i^0 \frac{\partial \widetilde{x}^k}{\partial x^i} = \mathcal{K}_i^0 \tag{1.5}$$

and analogue with (1.3) and (1.5) for $\mathcal{H}_i^j$ and $\mathcal{H}_i^0$ .In consequence we will take $\mathcal{H}_i^j = \mathcal{M}_i^j$ and $\mathcal{H}_i^0 = \mathcal{K}_i^0$ in the following.

Giving a nonlinear connection on E is obtaining the next adapted local base for $T_u E$: $\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^i}, \frac{\partial}{\partial z^i}, \frac{\partial}{\partial t} \right\}$ that is changing as the vectors as it results from(1.2) if there are verified the conditions (1.3) ,(1.4) ,(1.5) .

Considering a nonlinear connection fixed on E , we name *d-tensor of* (r,s) *type* a real function $t_{j_1.....j_s}^{i_1.....i_r}(x,y,z,t)$ that is changing after rule:

$$\widetilde{t}_{k_1....k_s}^{h_1....h_r}(\widetilde{u}) = \frac{\partial \widetilde{x}^{h_1}}{\partial x^{i_1}} .... \frac{\partial \widetilde{x}^{h_r}}{\partial x^{i_r}} \cdot \frac{\partial x^{j_1}}{\partial \widetilde{x}^{k_1}} .... \frac{\partial x^{j_s}}{\partial \widetilde{x}^{k_s}} t_{j_1....j_s}^{i_1....i_r}(u). \tag{1.6}$$

On E we can introduce relatively to the given nonlinear connection , the following geometrical structures.

$$F_j^i = dx^i \otimes \frac{\delta}{\delta y^j} + \delta y^i \otimes \frac{\partial}{\partial z^j} + \delta t \otimes \frac{\partial}{\partial t} \tag{1.7}$$

and his dual

$$F_j^{*i} = \delta y^i \otimes \frac{\delta}{\delta x^j} + \delta z^i \otimes \frac{\delta}{\delta y^j} + \delta t \otimes \frac{\partial}{\partial t}. \tag{1.7'}$$

The triplet $(F, \frac{\partial}{\partial t}, \delta t)$ verifies the conditions : $F^3 = \delta t \otimes \frac{\partial}{\partial t}$, $\delta t(\frac{\partial}{\partial t}) = 1$ and $rank\ F = 2n+1$ and it is named the cotangent structure of second order ([12])

Structure $\varphi = F - F^3$ it is an almost tangent of second order structure on E ([12]) , $rank\ \varphi = 2n.$

The triplet $(F^*, \frac{\partial}{\partial t}, \delta t)$ it is also a cotangent structure of second order named adjoint to F .

Analogue $\varphi^* = F^* - F^3$ it is a tangent structure of second order adjoint to φ .Easily there can be deduced links between these structures ([6]) ..

2. **Linear d-connections on E**

Let $E = T^2M \times R$ be the reonom bundle of second order endowed with a nonlinear connection conveniently chosen $N\Gamma = (\mathcal{M}_j^i, \mathcal{N}_j^i, \mathcal{K}_j^i)$ that determines the $TE = \mathcal{V}E \otimes \mathcal{H}E \otimes \mathcal{N}E$ decomposition, with the corresponding projectors .A field $X \in \mathcal{X}(E)$ will be decomposed in $X = vX + hX + nX.$

Definition 2. It is named *linear d-connection* on E a D linear connection on E that preserves trough parallelism the distributions $\mathcal{V}E, \mathcal{H}E, \mathcal{N}E$.

Theorem 1. *A linear connection D on E is a d-connection if and only if there are verified one of the following conditions :*

a) $(v+h)D_X nY = 0$, $(v+n)D_X hY = 0$, $(h+n)D_X vY = 0$

b) $D_X Y = vD_X vY + hD_X hY + nD_X nY$

c) $Dv = Dh = Dn = 0$

d) $DP_1 = 0, DP_2 = 0$ $DP_3 = 0$ *where* $P_1 = (n+h)-v$,$P_2 = (n+v)-h$, $P_3 = (v+h)-n$ *there are almost product structure on* E.

The proof results from the fact that: $D_X nY \in \mathcal{N}E$, $D_X hY \in \mathcal{H}E$, $D_X vY \in \mathcal{V}E$.

Because D is a R-linear application that can be extended to the whole d-tensors algebra, it results that :

Proposition 2. *It is only one operator of covariant derivation* D^n_X *named normal derivation thus that :*

$$D^n_X Y = D_{nX} Y \text{ and } D^n_X f = (nx)f : \ \forall X, Y \in \mathcal{X}(E),\ f \in \mathcal{F}(E). \tag{2.1}$$

Locally D^n can be expressed the following way :

$$D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = \overset{1}{L}{}^i_{jk} \frac{\delta}{\delta x^i}$$

$$D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^j} = \overset{2}{L}{}^i_{jk} \frac{\delta}{\delta y^i} \tag{2.2}$$

$$D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^j} = \overset{3}{L}{}^i_{jk} \frac{\partial}{\partial z^i} + L^0_{jk} \frac{\partial}{\partial t} \quad ; \quad D^n_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial t} = L^0_{0k} \frac{\partial}{\partial z^i} + L^0_{ok} \frac{\partial}{\partial t}$$

Analogous it is defined the D^h covariant $h-$derivation with the following local expressions.

$$D^h_{\frac{\delta}{\delta y^k}} \frac{\delta}{\delta x^j} = \overset{1}{F}{}^i_{jk} \frac{\delta}{\delta x^i} \quad ; \quad D^h_{\frac{\delta}{\delta z^k}} \frac{\partial}{\partial z^j} = \overset{3}{F}{}^i_{jk} \frac{\partial}{\partial z^i} + F^0_{jk} \frac{\partial}{\partial t} \tag{2.3}$$

$$D^h_{\frac{\delta}{\delta y^k}} \frac{\delta}{\delta y^j} = \overset{2}{F}{}^i_{jk} \frac{\delta}{\delta y^i} \quad ; \quad D^h_{\frac{\delta}{\delta y^k}} \frac{\partial}{\partial t} = F^i_{0k} \frac{\partial}{\partial z^i} + F^0_{0k} \frac{\partial}{\partial t}$$

and in totally the same way it is introduced D^v covariant $v-$derivation with local expressions

$$D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\delta}{\delta x^j} = \overset{1}{C}{}^{\,i}_{jk} \frac{\delta}{\delta x^i} \quad ; \quad D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\partial}{\partial t} = C^i_{0k} \frac{\partial}{\partial z^i} + C^0_{0k} \frac{\partial}{\partial t}$$

$$D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\delta}{\delta y^j} = \overset{2}{C}{}^{\,i}_{jk} \frac{\delta}{\delta y^i} \quad ; \quad D^v_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = C^0_{00} \frac{\partial}{\partial t} \tag{2.4}$$

$$D^v_{\frac{\partial}{\partial z^k}} \cdot \frac{\partial}{\partial z^j} = \overset{3}{C}{}^{\,i}_{jk} \frac{\partial}{\partial z^i}$$

The curvatures and torsions of a linear $d-$connection are written and are finding their local expressions through the direct calculation.([6])

3. M-linear connection on E

Let D be a linear $d-$ connection on E with local coefficients given by (2.1);(2.2);(2.3).

Definition 3. A $d-$ linear connection D on E it is said that it is a $M-$ *linear connection (Miron -connection)* if:

$$\overset{1}{L}{}^{i}_{jk} = \overset{2}{L}{}^{\,i}_{jk} = \overset{3}{L}{}^{\,i}_{jk};\ \overset{1}{F}{}^{i}_{jk} = \overset{2}{F}{}^{i}_{jk} = \overset{3}{F}{}^{i}_{jk};\ \overset{1}{C}{}^{i}_{jk} = \overset{2}{C}{}^{\,i}_{jk} = \overset{3}{C} \tag{3.1}$$

Let F and φ the almost cotangent structures of second order and respectively second order tangent locally given by (1.7) and $\varphi = F - F^3$,and (F^*, φ^*) their adjoint structures:

Definition 4. a) A D- linear connection on E is a F*-linear connection(respectively* F^*) if $D = 0$ and $D\dfrac{\partial}{\partial t} = 0$ (respectively $DF^* = 0, D\dfrac{\partial}{\partial t} = 0$).

b) A $D-$ linear connection on E is a *(φ, φ^*) -linear connection* on E if $DF = DF^* = 0$ and $D\dfrac{\partial}{\partial t} = 0$

c) A $D-$ linear connection on E is a $\varphi-$*linear connection* (respectively φ^*- linear connection) if $D\varphi = 0$ (respectively $D\varphi^* = 0$)

d) A $D-$ linear connection on E is a (φ, φ^*) $-$ *linear connection* if $D\varphi = D\varphi^* = 0$

Proposition 3. *A D -linear connection on E is a (F, F^*) -linear connection if and only if is a (φ, φ^*) -linear connection.*

Proof. From $DF = 0 \Rightarrow DF^3 = 0 \Rightarrow D(F - F^3) = 0 \Rightarrow D\varphi = 0$ and from $DF^* = 0 \Rightarrow D(F - F^*) = 0 \Rightarrow D\varphi^* = 0$. Reciprocal , we have $\varphi^* F^3 = 0$ and $F^3\varphi^* = 0$ (taking into account that $F^3(X_u) \in \mathcal{V}_u E)$ .It results that $DF^3 = 0$ and together with $D\varphi = 0$, $D\varphi^* = 0$ we are obtaining that $D(\varphi + F^3) = DF = 0$ and $(D\varphi^* + F^3) = DF^* = 0$.

Proposition 4. *A $(F.F^*)$-linear connection is a $d-$linear connection on E (F, F^*) .*

Proof:Is a $(F, F^*)-$ linear connection is a $(\varphi, \varphi^*)-$linear connection and using the fact that $v = \varphi^2\varphi^{*2}$,$\varphi^{*2} = n$ and $\varphi^*\varphi - \varphi^{*2}\varphi = h$ it results that $Dn = Dh = Dh = 0$ is a $d-$ linear connection on E .

Theorem 5. *A D linear connection on E is a M $-$linear connection if and only if is a (F, F^*)-linear connection .*

Proof:From the proposition 3.2 it results that if D is a (F, F^*)-linear connection than it is also a d-linear connection.

Because

$$(D_{\frac{\delta}{\delta x^k}} F)(\frac{\delta}{\delta x^j}) = (D^n_{\frac{\delta}{\delta x^k}} F)(\frac{\delta}{\delta x^j}) = (D^n_{\frac{\delta}{\delta x^k}} F)(\frac{\delta}{\delta x^j}) - FD^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} =$$

$$= D^n_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta y^j} - \overset{3}{L}{}^i_{jk} F(\frac{\delta}{\delta x^i}) = (\overset{2}{L}{}^i_{jk} - \overset{3}{L}{}^i_{jk})\frac{\delta}{\delta y^i}.$$

We are obtaining that $(D_{\frac{\delta}{\delta x^k}})(\frac{\delta}{\delta x^j}) = 0 \Leftrightarrow \overset{2}{L} = \overset{3}{L}$ .In an analogue way, taking these values of the adapted base fields, yields that $DF = DF^* = 0$,and hence D is a $M-$ linear connection on E.

We are waking the notifications $F^3 = p$ and $q = I - p$.

Theorem 6. *There exists $M-$linear connections on E. One of them is given by :*

$$\overset{B}{D}_X Y = \overset{B}{D}_{qX} qY + \overset{B}{D}_{qX} pY + \overset{B}{D}_{pX} qY + \overset{B}{D}_{pX} pY \tag{3.2}$$

where:

$$\overset{B}{D}_{qX} qY = \varphi^2 \left[\left(v + \frac{h}{2} \right) X, \varphi^{*2} y \right] + v \left[\left(n + \frac{h}{2} \right) X, vY \right] +$$

$$\varphi n\left[\left(n+\frac{h}{2}\right)X,\varphi^{*}hX\right]+\ \varphi^{*}v\left[\left(n+\frac{h}{2}\right)X,\varphi hY\right]+\varphi^{*2}\left[\left(n+\frac{h}{2}\right)X,nY\right]$$

$$\overset{B}{D}_{qX}\ pY=p\left[qX\ ,\ pY\right] \tag{3.3}$$

$$\overset{B}{D}_{pX}\ qY=\frac{1}{2}\{\varphi^{2}\left[pX\ ,\varphi^{2}Y\right]+\varphi^{*2}\left[pX\ ,\varphi^{2}Y\right]+(\frac{h}{2}+n)\left[pX\ ,(v+\frac{h}{2})Y\right]\}+$$

$$+\ \frac{1}{4}\{\varphi n\left[pX\ ,\varphi^{*}hY\right]+\varphi^{*}v\left[pX\ ,hY\right].\}$$

$$\overset{B}{D}_{pX}\ pY=\overset{0}{\nabla}_{pX}\ pY-\delta t(X)\delta t(Y)\ \overset{0}{\nabla}_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} \tag{3.4}$$

and $\overset{0}{\nabla}$ is a linear connection on E .

Proof. Trough the direct calculation it is verified that D is a linear connection and that $D\varphi=D\varphi^{*}=0$, so D is a $M-$ linear connection.

Given to X and Y values of the adapted base, from(3.3) results :

Corollary 7. The following functions on E

$$L_{ij}^{k}=\frac{\partial\mathcal{M}_{j}^{l}}{\partial z^{i}}\mathcal{M}_{l}^{k}+\frac{\partial\mathcal{N}_{j}^{k}}{\partial z^{i}}\ ;\qquad F_{ij}^{k}=\frac{\partial\mathcal{M}_{j}^{k}}{\partial z^{i}}\ ;\qquad C_{ij}^{k}=0$$

$$L_{i0}^{k}=L_{0j}^{k}=F_{0j}^{k}=C_{i0}^{0}=C_{0j}^{0}=C_{00}^{0}=0\ . \tag{3.5}$$

defining the coefficients of a M $-$linear connection on E ,named *Berwald connection* in the reonom bundle of second order

An interesting problem is the determination of the $M-$linear connection compatible with respect to a given metric structure on E . We will approach this in a coming paper .

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