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INVARIANT SETS OF RANDOM VARIABLES IN COMPLETE METRIC SPACES

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1. **Introduction**

The most known fractals are invariant sets with respect to a system of contraction maps, especially the so called self-similar sets. In a famous work, Hutchinson [6] first studied systematically the invariant sets in a general framework. He proved among others the following: *Let X be a complete metric space and $f_1, \dots, f_m : X \to X$ be contraction maps. Then there exists a unique compact set $K \subseteq X$ such that $K = \bigcup_{i=1}^{m} f_i(K)$.*

If the maps f_i are similitudes, this invariant set K is said to be *self-similar*.

Our aim in this work is to generalize the above theorem of Hutchinson for random variables in complete metric spaces using some results from the theory of probabilistic metric spaces.

The theory of probabilistic metric spaces, introduced in 1942 by K. Menger [11], was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [14]. The study of contraction mappings for probabilistic metric spaces was initiated by V. M. Sehgal [16],[17], and H. Sherwood [19].

Falconner [4],Graf [5], and Hutchinson and Rüschendorf [6] used contraction methods to obtain random self-similar fractal sets by essential applying ordinary metrics to a.e. realization in the random setting. The same ideas were used by Arbeiter[1], Olsen [12], and Hutchinson and Rüschendorf [7],[8],[9], to obtain random self similar fractal measures. In these works a finite first moment condition of the distance function is essential. Using probabilistic metric space techniques, we can weak this first moment condition, as will be shown for fractal sets in Section 4.

2. **Preliminaries**

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$. A mapping $F : \mathbf{R} \to [0,1]$ is called a *distribution function* if it is non-decreasing, left continuous with $\inf F = 0$.(see [2]) By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$": $F \leq G$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Throughout this paper H will denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x,y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbf{R}$ by $F_{x,y}(t)$, respectively. The pair $(X, \mathcal{F})$ is a *probabilistic metric space* (briefly *PM space*) if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbf{R}$;

2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0. if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s+t) = 1$.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a, 1) = a$ for every $a \in [0,1]$;

5^0. $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$

6^0. if $a \geq c$ and $b \geq d$ then $T(a,b) \geq T(c,d)$;

7^0. $T(a, T(b,c)) = T(T(a,b), c)$ for every $a, b, c \in [0,1]$.

A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, where T is a t-norm, and instead of 3^0 we have the stronger condition

8^0. $F_{x,y}(s+t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbf{R}_+$.

The (t, ϵ)-topology in a Menger space was introduced in 1960 by B. Schweizer and A. Sklar [13]. The base for the neighbourhoods of an element $x \in X$ is given by

$$\{U_x(t,\epsilon) \subseteq X : t > 0, \epsilon \in]0,1[\},$$

where

$$U_x(t,\epsilon) := \{y \in X : F_{x,y}(t) > 1-\epsilon\}.$$

If the t-norm T satisfies the condition

$$sup\{T(a,a) : a \in [0,1[\} = 1,$$

then the (t,ϵ) -topology is metrizable (see [15]).

In 1966, V.M. Sehgal [16] introduced the notion of a contraction mapping in PM spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0,1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbf{R}_+$.

A sequence $(x_n)_{n\in\mathbf{N}}$ from X is said to be *fundamental* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 0$. The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

Set

$$\mathcal{D}^+ = \{F \in \Delta^+ : \sup_{t\in R} F(t) = 1\}.$$

In the following we always suppose that $(X, \mathcal{F}, T)$ is a Menger space with $\mathcal{F} : X \times X \to \mathcal{D}^+$ and T is continuous.

Let A be a nonempty subset of X. The function $D_A : \mathbf{R} \to [\mathbf{0},\mathbf{1}]$ defined by

$$D_A(t) := \sup_{s<t} \inf_{x,y\in A} F_{x,y}(s)$$

is called the *probabilistic diameter of A.* It is easy to check that $D_A \in \Delta^+$. The set $A \subseteq X$ is *probabilistic bounded* if $D_A \in \mathcal{D}^+$. If B and C are two subsets of X with $B \cap C \neq \emptyset$, then

$$D_{B \cup C}(s+t) \geq T(D_B(s), D_C(t)),\ s,t \in \mathbf{R} \tag{1}$$

(see [3, Theorem 10]). In particular, every finite subset of X is probabilistic bounded.

We also define the *probabilistic radius* $E_A : \mathbf{R} \to [0,1]$ of the set A:

$$E_A(t) := \sup_{s<t} \sup_{y \in A} \inf_{x \in A} F_{x,y}(s).$$

By definition it is easy to verify the following property:

Lemma 2.1.

$$E_A(t) \geq D_A(t),$$

and

$$D_A(2t) \geq T(E_A(t), E_A(t)),\ \textit{for all}\ t > 0.$$

Let A and B nonempty subsets of X. The *probabilistic Hausdorff-Pompeiu distance* between A and B is the function $F_{A,B} : \mathbf{R} \to [\mathbf{0}, \mathbf{1}]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T(\inf_{x \in A} \sup_{y \in B} F_{x,y}(s), \inf_{y \in B} \sup_{x \in A} F_{x,y}(s)).$$

Lemma 2.2. *For the nonempty subsets A and B of X we have*

$$E_A(t_1 + 2t_2) \geq T(D_B(t_1), F_{A,B}(t_2))\ \textit{for all}\ \ t_1, t_2 > 0.$$

Proof. Let $x, y \in A$, $z, u \in B$ and $s_1, s_2 > 0$. By 8^0 we have

$$F_{x,y}(s_1 + 2s_2) \geq T(F_{x,z}(s_1 + s_2), F_{z,y}(s_2)) \geq$$

$$\geq T(T(F_{x,u}(s_2), F_{u,z}(s_1)), F_{y,z}(s_2)) \geq T(T(F_{x,u}(s_2), D_B(s_1)), F_{y,z}(s_2)) =$$

$$= T(D_B(s_1)), T(F_{x,u}(s_2), F_{y,z}(s_2))).$$

Simple calculations show

$$\sup_{y \in A} \inf_{x \in A} F_{x,y}(s_1 + 2s_2) \geq T(D_B(s_1), T(\inf_{x \in A} \sup_{u \in B} F_{x,u}(s_2), \inf_{z \in B} \sup_{y \in A} F_{y,z}(s_2))).$$

If we take the supremum by $s_1 < t_1$ and $s_2 < t_2$ we obtain the required inequality. □

Proposition 2.1. *If $\mathcal{C}$ is a nonempty collection of nonempty closed bounded sets in a Menger space $(X, \mathcal{F}, T)$ with T continuous, then $(\mathcal{C}, F_C, T)$ is also Menger space, where $\mathcal{F}_C$ is defined by $\mathcal{F}_C(A, B) := F_{A,B}$ for all $A, B \in \mathcal{C}$.*

Proof. See [3],[10]. □

Proposition 2.2. *Let $T_m(a, b) := \max\{a + b - 1, 0\}$. If $(X, \mathcal{F}, T_m)$ is a complete Menger space and $\mathcal{C}$ is the collection of all nonempty closed bounded subsets of X in $(t, \epsilon)-$ topology, then $(\mathcal{C}, \mathcal{F}_C, T_m)$ is also a complete Menger space.*

Proof. Let $(A_n)_{n\in\mathbf{N}}$ be a fundamental sequence in $\mathcal{C}$ and let

$$A = \{x \in X : \forall n \in \mathbf{N}, \exists x_n \in A_n, \forall t > 0, \lim_{n\to\infty} F_{x_n,x}(t) = 1\}. \tag{2}$$

Let $\overline{A}$ denote the closure of A. By [3, Theorem 15] we have $F_{A_n,A} = F_{A_n,\overline{A}}$, so it is enough to show that (i) $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$, and (ii) $\overline{A} \in \mathcal{C}$.

(i) Let $t > 0$ and $\epsilon > 0$ be given. Then there exists $n_\epsilon(t) \in \mathbf{N}$ such that $n, m > n_\epsilon$ implies $F_{A_n,A_m}(\frac{t}{4}) > 1-\frac{\epsilon}{4}$. Let $n > n_\epsilon(t)$. We claim that $F_{A_n,A}(t) \geq 1-\epsilon$.

If $x \in A$, then there is a sequence $(x_k)_{k\in\mathbf{N}}$ with $x_k \in A_k$ and $\lim_{k\to\infty} F_{x_k,x}(\frac{t}{4}) = 1$. So, for large enough $k > n_\epsilon(t)$, we have $F_{x_k,x}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. Since $F_{A_n,A_k}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$, there exist $y \in A_n$ such that $F_{x_k,y}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. By 8^0 we have $F_{x,y}(\frac{t}{2}) > 1 - \frac{\epsilon}{4}$, hence

$$\sup_{s<t} \inf_{x\in A} \sup_{y\in A_n} F_{x,y}(s) > 1 - \frac{\epsilon}{2}. \tag{3}$$

Now suppose that $y \in A_n$ is arbitrary. Choose integers $k_1 < k_2 < ... < k_i < ...$ so that $k_1 = n$ and

$$F_{A_k,A_{k_i}}(\frac{t}{2^{i+2}}) > 1 - \frac{\epsilon}{2^{i+2}},$$

for all $k > k_i$. We have $\inf_{z\in A_{k_i}} \sup_{x\in A_k} F_{x,z}(\frac{t}{2^{i+1}}) > 1-\frac{\epsilon}{2^{i+2}}$. Then define a sequence (y_k) with $y_k \in A_k$ as follows. For $k < n$, let $y_k \in A_k$be arbitrarily and $y_n = y$. If y_{k_i} has been chosen and $k_i < k \leq k_{i+1}$, take $y_k \in A_k$ with $F_{y_{k_i},y_k}(\frac{t}{2^{i+2}}) > 1 - \frac{\epsilon}{2^{i+2}}$. Then, for $k_i < k \leq k_{i+1} < ... < k_j < l \leq k_{j+1}$, we have

$$F_{y_l,y_k}(\frac{t}{2^i}) \geq F_{y_k,y_{k_i}}(\frac{t}{2^{i+1}}) + F_{y_{k_i},y_{k_{i+1}}}(\frac{t}{2^{i+2}}) + ... + F_{y_{k_{j-1}},y_{k_j}}(\frac{t}{2^{j+1}})+$$

$$+F_{y_{k_j},y_l}(\frac{t}{2^{j+1}}) - (j - i + 1) > 1 - \frac{\epsilon}{2^{i+1}}.$$

Let $0 < r$, $0 < \eta < 1$, and choose i so that $\frac{t}{2^i} < r$ and $\frac{\epsilon}{2^{i+1}} < \eta$. We have

$$F_{y_k,y_l}(r) \geq F_{y_k,y_l}(\frac{t}{2^i}) > 1 - \frac{\epsilon}{2^{i+1}} > 1 - \eta.$$

Hence (y_k) is a fundamental sequence, so it converges. Let x be its limit. Therefore $x \in A$, and we have

$$F_{x,y}(\frac{t}{2}) \geq F_{x,y_k}(\frac{t}{4}) + F_{y_k,y}(\frac{t}{4}) - 1.$$

Select $k > n$ such that $F_{x,y_k}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$. Since $F_{y,y_k}(\frac{t}{4}) > 1 - \frac{\epsilon}{4}$, it follows that $F_{x,y}(\frac{t}{2}) > 1 - \frac{\epsilon}{2}$. Therefore we have

$$\sup_{s<t} \inf_{y \in A_n} \sup_{x \in A} F_{x,y}(s) > 1 - \frac{\epsilon}{2}. \tag{4}$$

By (3), the lather implies

$$F_{A_n,A}(t) = \sup_{s<t} T_m(\inf_{x \in A} \sup_{y \in A_n} F_{x,y}(s), \inf_{y \in A_n} \sup_{x \in A} F_{x,y}(s)) > 1 - \epsilon.$$

Thus $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $t > 0$,hence part (i) is proved..

(ii) Taking $\epsilon = 1$ in the last argument, we have proved that A is nonempty.

Next we have to show that A is bounded. Since $\lim_{n\to\infty} F_{A_n,A}(t) = 1$, for all $\epsilon > 0$ and $t_0 > 0$ there exists $n_0 \in N$ such that, for every $n > n_0$, we have $\inf_{x\in A} \sup_{w\in A_n} F_{x,w}(t_0) > 1-\epsilon$ and $\inf_{y\in A_n} \sup_{x\in A} F_{x,y}(t_0) > 1-\epsilon$. The set A_n being probabilistic bounded, for all $\epsilon > 0$ there is $t_\epsilon > t_0$ such that $\inf_{u,v\in A_n} F_{u,v}(t_\epsilon) > 1-\epsilon$.

On the other hand, $x, y \in A$ there exist $u, v \in A_n$ such that

$$F_{x,u}(t_0) > 1 - \epsilon,\ F_{y,v}(t_0) > 1 - \epsilon.$$

We have

$$F_{x,y}(3t_\epsilon) \geq T_m(F_{x,u}(t_\epsilon), F_{u,y}(2t_\epsilon)) \geq T_m(F_{x,u}(t_0), T_m(F_{u,v}(t_\epsilon), F_{v,y}(t_0))) > 1 - 3\epsilon.$$

Therefore $D_A(3t_\epsilon) \geq 1 - 3\epsilon$, consequently we have $\sup_{t\in\mathbf{R}} D_A(t) = 1$. By [3], it follows that $D_A = D_{\overline{A}}$, hence $\overline{A} \in \mathcal{C}$. □

3. **Invariant sets in E-spaces**

The notion of E-space was introduced by Sherwood [20] in 1969. Next we recall this definition. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (M, d) be a metric space. The ordered pair $(\mathcal{E}, F)$ is an *E-space over the metric space* (M, d) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into M and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega | \; d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbf{R}$. Usually $(\Omega, \mathcal{K}, P)$ is called the base and (M, d) the target space of the E-space. If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \neq H, \quad for \;\; x \neq y,$$

with H defined in section 2, then $(\mathcal{E}, \mathcal{F})$ is said to be a *canonical E-space.* H. Sherwood [20] proved that every canonical $\mathcal{E}$-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$. In the following we suppose that E is a canonical E-space.

The convergence in an $\mathcal{E}$-space is exactly the probability convergence. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

Proposition 3.1. *If (M,d) is a complete metric space then the E-space* $(\mathcal{E}, F)$ *is also complete.*

Proof. This property is well-known if $M = R$ (see e.g. [21, Theorem VII.4.2.]). In the general case the proof is analogous and we omit it. □

Proposition 3.2. *If* A *is a nonempty probabilistic bounded subset of* $\mathcal{E}$ *and* $f : \mathcal{E} \to \mathcal{E}$ *is a contraction with ratio* r *then f(A) is also probabilistic bounded, where*

$$f(A) = \{f(x) \,|\, x \in A\}.$$

Proof. We have

$$D_{f(A)}(t) = \sup_{s<t} \inf_{u,v \in f(A)} F_{u,v}(s) =$$

$$= \sup_{s<t} \inf_{x,y \in A} P(\{\omega \in \Omega | \; d(f(x)(\omega), f(y)(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{x,y \in A} P(\{\omega \in \Omega | \; d(x(\omega), y(\omega)) < \frac{s}{r}\}) \geq$$

$$\geq \sup_{s<t} \inf_{x,y \in A} F_{x,y}(s) = D_A(t).$$

Since $\sup_{t>0} D_A(t) = 1$, it follows that $\sup_{t>0} D_{f(A)}(t) = 1$. □

The main result of this paper is the following:

Theorem 3.1. *Let $(\mathcal{E}, F)$ be a complete E- space, $N \in \mathbf{N}^*$,, and let $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ be contractions with ratio $r_1, ...r_N$, respectively. Suppose that there exists an element $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega | d(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{5}$$

for all $i \in \{1, .., N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded subset K of $\mathcal{E}$ such that

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Proof. Let $\Phi : 2^{\mathcal{E}} \to 2^{\mathcal{E}}$ be defined by

$$\Phi(A) := f_1(A) \cup f_2(A) \cup ... \cup f_N(A).$$

Let $A_0 = \{z\}$ and $A_n = \Phi(A_{n-1})$ for $n \geq 1$. Let $r = \max\{r_1, ..., r_N\}$, J be the finite alphabet $\{1, .., N\}$, and, for $\sigma = \sigma_1...\sigma_n \in J^n$, set $f_\sigma = f_{\sigma_1} \circ f_{\sigma_2} \circ ... \circ f_{\sigma_n}$. We have:

$$A_n = \cup_{\sigma \in J^n} f_\sigma(A_0).$$

First we show that $(A_n)_{n \in N}$ is a fundamental sequence in $(\mathcal{C}, F_C, T_m)$.

Since $A_{n+k} = \Phi^n(A_k)$ and $A_n = \Phi^n(A_0)$, we have

$$\inf_{u \in A_n} \sup_{v \in A_{k+n}} F_{u,v}(s) = \inf_{u \in \cup_{\sigma \in J^n} f_\sigma(A_0)} \sup_{v \in \cup_{\sigma \in J^n} f_\sigma(A_k)} F_{u,v}(s).$$

Observe, there exists $\sigma' \in J^n$ such that

$$\inf_{u \in A_n} \sup_{v \in A_{k+n}} F_{u,v}(s) = \inf_{u \in f_{\sigma'}(A_0)} \sup_{v \in \cup_{\sigma \in J^n} f_\sigma(A_k)} F_{u,v}(s) \geq$$

$$\geq \inf_{u \in f_{\sigma'}(A_0)} \sup_{v \in f_{\sigma'} A_k} F_{u,v}(s) = \inf_{x \in A_0} \sup_{y \in A_k} F_{f_{\sigma'}(x), f_{\sigma'}(y)}(s) \geq$$

$$\geq \sup_{y \in A_k} P(\{\omega \in \Omega | \ r^n d(z(\omega), y(\omega)) < s\}) =$$

$$= \max_{y \in \cup_{\tau \in J^k} f_\tau(A_0)} P(\{\omega \in \Omega | \ r^n d(z(\omega), y(\omega)) < s\}) \geq$$

$$\geq \max_{y \in \cup_{\tau \in J^k} f_\tau(A_0)} P(\{\omega \in \Omega | \ r^n d(z(\omega), y(\omega)) < s \cdot (1 + \sqrt{r} + \cdots + \sqrt{r}^{k-1})(1 - \sqrt{r})\}) \geq$$

$$\geq \max_{\tau \in J^k} P(\{\omega \in \Omega | \ r^n [d(z(\omega), f_{\tau_1}(z(\omega))) + d(f_{\tau_1}(z(\omega)), f_{\tau_1 \tau_2}(z(\omega))) + \cdots$$

$$\cdots + d(f_{\tau_1 \cdots \tau_{k-1}}(z(\omega)), f_{\tau_1 \cdots \tau_k}(z(\omega)))] < s \cdot (1 + \sqrt{r} + \cdots + \sqrt{r}^{k-1})(1 - \sqrt{r})\}) \geq$$

$$\geq \max_{\tau\in J^k}[P(\{\omega\in\Omega|\ d(z(\omega),f_{\tau_1}(z(\omega)))<\frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ d(f_{\tau_1}(z(\omega)),f_{\tau_1\tau_2}(z(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\})+\cdots$$

$$\cdots P(\{\omega\in\Omega|\ d(f_{\tau_1\ldots\tau_{k-1}}(z(\omega)),f_{\tau_1\ldots\tau_k}(z(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})]-(k-1)\geq$$

$$\max_{\tau\in J^k}[P(\{\omega\in\Omega|\ d(z(\omega),f_{\tau_1}(z(\omega)))<\frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ rd(z(\omega)),f_{\tau_2}(z(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\})+\cdots+$$

$$+P(\{\omega\in\Omega|\ r^{k-1}d(z(\omega),f_{\tau_m}(z(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})]-(k-1)=$$

$$=1-\min_{\tau\in J^m}[P(\{\omega\in\Omega|\ d(z(\omega),f_{\tau_1}(z(\omega)))\geq\frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ d(z(\omega),f_{\tau_2}(z(\omega)))\geq\frac{s(1-\sqrt{r})\sqrt{r}}{r^{n+1}}\})+\cdots+$$

$$+P(\{\omega\in\Omega|\ d(z(\omega),f_{\tau_k}(z(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^{n+k-1}}\})]\geq$$

$$\geq 1-\gamma\cdot r^n\left(\frac{1}{s(1-\sqrt{r})}+\frac{r^{1/2}}{s(1-\sqrt{r})}+...+\frac{r^{(k-1)/2}}{s(1-\sqrt{r})}\right)>$$

$$>1-\gamma\frac{r^n}{s(1-\sqrt{r})^2}.$$

Since

$$\lim_{n\to\infty}\left(1-\gamma\frac{r^n}{s(1-\sqrt{r})^2}\right)=1,$$

we have, for $t>0$,

$$\lim_{n\to\infty}F_{A_n,A_{k+n}}(t)=1,$$

uniformly with respect to k. The space $(\mathcal{E},F)$ being complete, (A_n) is convergent. Let K be its limit.

Next we show that K is a fixed point of Φ. For $i\in\{1,...,N\}$, $x\in A_{n-1}, y\in K$ and $s>0$, we have

$$F_{f_i(x),f_i(y)}(s)\geq F_{x,y}(s).$$

There exists $i\in J$ such that

$$\inf_{u\in\Phi(A_{n-1})}\sup_{v\in\Phi(K)}F_{u,v}(s)=\inf_{u\in f_i(A_{n-1})}\sup_{v\in\Phi(K)}F_{u,v}(s)\geq$$

$$\geq\inf_{x\in A_{n-1}}\sup_{y\in K}F_{f_i(x),f_i(y)}(s)\geq\inf_{x\in A_{n-1}}\sup_{y\in K}F_{x,y}(s).$$

In a similar way

$$\inf_{v\in\Phi(K)}\sup_{u\in\Phi(A_{n-1})}F_{u,v}(s)\geq\inf_{y\in K}\sup_{x\in A_{n-1}}F_{x,y}(s).$$

Then it follows

$$F_{A_n,\Phi(K)}(\frac{t}{2})\geq F_{A_{n-1},K}(\frac{t}{2})\ \text{ for all } t>0.$$

Using 8^0 one obtains

$$F_{K,\Phi(K)}(t)\geq T_m(F_{K,A_n}(\frac{t}{2}),F_{A_n,\Phi(K)}(\frac{t}{2}))\geq T_m(F_{K,A_n}(\frac{t}{2}),F_{A_{n-1},K}(\frac{t}{2})).$$

Since $\lim_{n\to\infty}A_n=K$, we have

$$F_{K,\Phi(K)}(t)=1\ \ for\ all\ \ t>0,$$

therefore

$$K=\Phi(K).$$

For the uniqueness we suppose that there exist closed and bounded K and K' such that $\Phi(K)=K$ and $\Phi(K')=K'$. For $x\in K$, $y\in K'$, $\sigma\in J^n$, and $s>0$, we have

$$F_{f_\sigma(x),f_\sigma(y)}(s)\geq F_{x,y}(\frac{s}{r^n}).$$

Let $\sigma'\in J^n$ be such that

$$\inf_{v\in\cup_{\sigma\in J^n}f_\sigma(K')}\sup_{u\in\cup_{\sigma\in J^n}f_\sigma(K)}F_{v,u}(s)=\inf_{x\in f'_\sigma(K')}\sup_{u\in\cup_{\sigma\in J^n}f_\sigma(K)}F_{v,u}(s)\geq$$

$$\geq\inf_{v\in f_{\sigma'}(K')}\sup_{u\in f_{\sigma'}(K)}F_{v,u}(s)\geq\inf_{y\in K'}\sup_{x\in K}F_{x,y}(\frac{s}{r^n}).$$

Similarly,

$$\inf_{v\in\cup_{\sigma\in J^n}f_\sigma(K')}\sup_{u\in\cup_{\sigma\in J^n}f_\sigma(K)}F_{v,u}(s)\geq\inf_{x\in K}\sup_{y\in K'}F_{x,y}(\frac{s}{r^n}).$$

Since $K=\Phi^n(K)=\cup_{\sigma\in J^n}f_\sigma(K),\ K'=\Phi^n(K')=\cup_{\sigma\in J^n}f_\sigma(K')$, we have

$$F_{K,K'}(t)\geq F_{K,K'}(\frac{t}{r^n})\ \text{ for all } t>0.$$

Using $\lim_{n\to\infty}r^n=0$, we have

$$F_{KK'}(t)=1\ \text{ for all } t>0,$$

therefore $K=K'$. □

Corollary 3.1. *Let $(\mathcal{E}, F)$ be a complete E- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega |\ d(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \text{ for all } t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

Remark: If the mean values $\int_\Omega d(z(\omega), f_i(x(\omega)))dP$ for $i \in \{1, ..., N\}$ are finite, then by the Chebisev inequality, condition (5) is satisfied. However, the condition (5) can also be satisfied for $\int_\Omega d(z(\omega), f(z(\omega)))dP = \infty$. For example, let $\Omega =]0, 1]$ with the Lebesque measure and let $f(x) = \frac{x(\omega)}{3} + \frac{1}{\omega}$. Then for $z(\omega) \equiv 0$, the above expectation is ∞, but, for $\gamma = 1$, the condition (5) holds.

As in [6], the invariant set can be modeled by strings. Let $N \geq 1$, and define

$$\{1, ..., N\}^* = \cup_{k \in \mathbf{N}} \{1, ..., N\}^k.$$

If $\tau \in \{1, ..., N\}^*$, $\tau = \tau_1.\tau_2...\tau_k$, then $|\ \tau| = k$ is the length of τ. Set $f_\tau : \mathcal{E} \to \mathcal{E}$, $f_\tau := f_{\tau_1} \circ f_{\tau_2} \circ ... \circ f_{\tau_k}$. If $A \subset \mathcal{E}$, we set $A_{\tau_1...\tau_k} := f_\tau(A)$.

Let $\{1, ..., N\}^{\mathbf{N}}$ carry the product of the discrete topology on $\{1, ..., N\}$. For $\sigma \in \{1, ..., N\}^* \cup \{1, ..., N\}^{\mathbf{N}}$ with k$\leq |\ \sigma|$ let $\sigma_{|k} = \sigma_1.\sigma_2...\sigma_k$ be the restriction of σ to its first k entries.

Let K be the invariant set from Theorem 3. As in [6], we can show that

a) $K_{\sigma_1...\sigma_k} = \cup_{\sigma_{k+1}=1}^n K_{\sigma_1...\sigma_k\sigma_{k+1}}$

b) $K \supset K_{\sigma_1} \supset ... \supset K_{\sigma_1...\sigma_k} \supset ....$

Proposition 3.3. *Let the hypotheses of Theorem 3 be satisfied. Then, for all $t > 0$, we have*

$$lim_{k\to\infty} D_{f_{\sigma_{|k}}(K)}(t) = 1.$$

Proof. Let A_n be the set defined in the proof of Theorem 3. If f is an r-contraction, then $F_{f(A_n), f(K)}(t) \geq F_{A_n,K}(t)$ for $t > 0$. Let $\sigma \in \{1, ..., N\}^* \cup \{1, ..., N\}^{\mathbf{N}}$. Since $\lim_{n\to\infty} F_{A_n,K}(t) = 1$ for $t > 0$, it follows that

$$\lim_{n\to\infty} F_{f_{\sigma|k}(A_n), f_{\sigma|k}(K)}(t) = 1 \tag{6}$$

uniformly with respect to k.

We have

$$D_{f_{\sigma|k}(A_n)}(t) = \sup_{s<t} \inf_{x,y \in f_{\sigma|k}(A_n)} P(\{\omega \in \Omega|\ d(x(\omega), y(\omega)) < s\}) =$$

$$= \sup_{s<t} \inf_{u,v \in A_n} P(\{\omega \in \Omega|\ d(f_{\sigma_1 \ldots \sigma_k}(u)(\omega), f_{\sigma_1 \ldots \sigma_k}(v)(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{u,v \in A_n} P(\{\omega \in \Omega|\ r_{\sigma_1} ... r_{\sigma_k} d(u(\omega), v(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{u,v \in A_n} P(\{\omega \in \Omega|\ r^k d(u(\omega), v(\omega)) < s\}) \geq$$

$$\geq \sup_{s<t} \inf_{u,v \in A_n} [P(\{\omega \in \Omega|\ d(u(\omega), v(\omega)) < \frac{s}{2r * k}\}) +$$

$$+ P(\{\omega \in \Omega|\ d(z(\omega), v(\omega)) < \frac{s}{2r^k}\})] - 1 \geq$$

$$\geq 1 - \frac{\gamma}{(1-\sqrt{r})^2} \cdot r^n.$$

Hence

$$\lim_{k \to \infty} D_{f_{\sigma \rceil k}(A_n)}(t) = 1 \text{ for all } t > 0 \text{ and } n \in \mathbf{N}.$$

By Lemma 2.2 we have

$$D_{f_{\sigma|k}(K)}(t) \geq D_{f_{\sigma|k}(A_n)}(t) + F_{f_{\sigma|k}(A_n), f_{\sigma|k}(K)}(t) - 1.$$

Using (6) it follows the assertion. □

Proposition 3.4. *For all* $\sigma \in \{1, ..., N\}^{\mathbf{N}}$ *there exists a unique element* $x_\sigma \in \cap_{n \in N} \overline{K}_{\sigma_1 \ldots \sigma_n}$

Proof. For every $n \in \mathbf{N}$ we choose an element $x_n \in K_{\sigma_1 \ldots \sigma_n}$. Let $m < n$, then $x_m, x_n \in K_{\sigma_1 \ldots \sigma_m}$. Since

$$lim_{k \to \infty} D_{f_{\sigma|k}(K)}(t) = 1 \text{ for } t > 0, \text{ and } \epsilon > 0,$$

there exists $m_0 \in N$ such that, for all $m > m_0$,

$$\inf_{x,y \in K_{\sigma_1 \ldots \sigma_m}} P\{\omega \in \Omega | d(x(\omega), y(\omega)) < t\} > 1 - \epsilon.$$

It follows, for $m, n > m_0$, $P(\{\omega \in \Omega | d(x_n(\omega), x_m(\omega)) < t\}) > 1 - \epsilon$, therefore $(x_n)_{n \in N}$ is a Cauchy sequence. Since the space $(\mathcal{E}, \mathcal{F})$ is complete, it follows the convergence of $(x_n)_{n \in N}$. Let x_σ be its limit. Then $x_\sigma \in \cap_{n \in N} \overline{K}_{\sigma_1 \ldots \sigma_n}$.

Since $\lim_{n \to \infty} D_{\overline{K}_{\sigma_1 \ldots \sigma_n}}(t) = 1$ for all $t > 0$, it follows that x_σ is the unique element of the intersection. □

Proposition 3.5. *The map* $\pi : \{1, ..., N\}^{\mathbf{N}} \to K$ *given by* $\pi(\sigma) = x_\sigma$ *is a continuous map onto* K.

Proof. Let $\sigma = \sigma_1 ... \sigma_n ... \in \{1, ..., N\}^{\mathbf{N}}$ and let $\epsilon > 0$. Since $\pi(\sigma) = x_\sigma \in \cap_{n \in N} \overline{K}_{\sigma_1 ... \sigma_n}$ and $\lim_{n \to \infty} D_{\overline{K}_{\sigma_1 \cdots \sigma_n}}(t) = 1$ for all $t > 0$, there exists $n_0 \in N$ such that

$$D_{\overline{K}_{\sigma_1 \cdots \sigma_n}}(t) > 1 - \epsilon \text{ for all } n > n_0.$$

For $y \in \overline{K}_{\sigma_1 ... \sigma_n}$ we have

$$P(\{\omega \in \Omega | \; d(y, \pi(\sigma)) < t\}) > 1 - \epsilon,$$

hence $\overline{K}_{\sigma_1 \cdots \sigma_n} \subset U_{\pi(\sigma)}(t, \epsilon)$ for $n > n_0$. Since $\overline{K_{\sigma_1 \ldots \sigma_n}}$ contains the image of the open set $\{\beta | \beta_i = \sigma_i, if \; i \leq n\}$, it follows π is continuous.

Let $K' = \pi(\{1, ..., N\}^{\mathbf{N}})$. Observe $K' \subset K$ and K' is a compact set. We show that K' is an invariant set. If $y \in K'$, then there exists $\sigma \in \{1, ..., N\}^{\mathbf{N}}$ such that $y = \pi(\sigma) \in f_{\sigma_1}(K')$. So $K' \subset \cup_{i=1}^{l} f_i(K')$.

If $y \in \cup_{i=1}^{l} f_i(K')$ then there exists $j \in \{1, ..., l\}$ such that $y \in f_j(K')$, hence, for any $\sigma' \in \{1, ..., N\}^{\mathbf{N}}$, $\; y = f_j(\pi(\sigma')) = \pi(j\sigma') \in K'$.

Since the closed bounded invariant set is unique, it follows $K = K'$. □

Corollary 3.2. *The invariant set in Theorem 3 is compact.*

4. **Self similar fractal sets**

Recently Hutchinson and Rüschendorf [9] gave a simple proof for the existence and uniqueness of invariant random sets using the L^∞-metric. The underlying probability space for the iteration procedure is generated by selecting independent and identically distributed scaling laws. A *scaling law* $\mathbf{S}$ is an N-tuple $(S_1, ..., S_N)$, $N \geq 2$, of Lipschitz maps $S_i : \mathbf{R}^n \to \mathbf{R}^n$. Let $r_i = Lip S_i$. *A random scaling law* $\mathbf{S} = (S_1, S_2, ..., S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S} = dist\mathbf{S}$ for the probability distribution determined by $\mathbf{S}$ and $\stackrel{d}{=}$ for the equality in distribution.

If K is a random set, then the random set $\mathbf{S}K$ is defined (up to probability distribution) by

$$\mathbf{S}K = \cup_i S_i K^{(i)},$$

where $\mathbf{S}, K^{(1)}, ..., K^{(N)}$ are independent of one another and $K^{(i)} \stackrel{d}{=} K$.

We say K *satisfies the scaling law* $\mathbf{S}$, or is a *self-similar random fractal set,* if

$$\mathbf{S}K \stackrel{d}{=} K, \text{ or equivalently } \mathcal{S}K = K.$$

Let $\mathbf{C}$ be the set of random compact sets K such that

$$ess\sup_{\omega} d_{\mathcal{H}}(K^{\omega}, \delta_B^{\omega}) < \infty, \tag{7}$$

for some, and hence any, fixed compact set $B \subset R^n$, where $d_{\mathcal{H}}$ is the hausdorff metric. By δ_B we mean the random set equal a.s. to B. In [9], Hutchinson and Rüschendorf generate random sets in the following manner:

Beginning with a nonrandom set K_0 one defines a sequence of random sets

$$\mathbf{S}K_0 = \cup_i S_i K_0,$$

$$\mathbf{S}^2 K_0 = \cup_{i,j} S_i \circ S_j^i K_0,$$

$$\mathbf{S}^3 K_0 = \cup_{i,j,k} S_i \circ S_j^i \circ S_k^{ij} K_0,$$

etc.: where $\mathbf{S^i} = (S_1^i, S_2^i, ..., S_N^i)$, for $i \in \{1, ..., N\}$, are independent of each other and of $\mathbf{S}$, the $\mathbf{S}^{ij} = (S_1^{ij}, S_2^{ij}, ..., S_N^{ij})$, for $i, j \in \{1, ..., N\}$ are independent of each other and of $\mathbf{S}$ and $\mathbf{S^i}$, etc.

A *construction tree* (or a construction process) is a map $\omega : \{1, ..., N\}^* \to \Gamma$, where Γ is the set of (nonrandom) scaling laws. The sample space of all construction trees is denoted by $\tilde{\Omega}$. The underlying probability space $(\tilde{\Omega}, \tilde{\mathcal{K}}, \tilde{P})$ for the iteration procedure is generated by selecting identical distributed and independent scaling laws $\omega(\sigma) \stackrel{d}{=} \mathbf{S}$ for each $\sigma \in \{1, ..., N\}^*$ (see [9]). It is well known the following result:

Theorem 4.1. ([4],[5],[9]) *If* $\mathbf{S} = (S_1, S_2, ..., S_N)$ *is a random scaling law with*

$$\lambda := ess\sup_{\omega}\ r^{\omega} < 1 \tag{8}$$

(where $r^{\omega} = \max_i LipS_i^{\omega}$*), then for any (nonrandom) compact set* K_0*,*

$$ess\sup_{\omega}\ d_{\mathcal{H}}(\mathbf{S}K_0, K^*) \leq \frac{\lambda^k}{1-\lambda} ess\sup_{\omega}\ d_{\mathcal{H}}(K_0, \mathbf{S}K_0) \to 0$$

as $k \to \infty$*, where* K^* *does not depend on* K_0*. In particular,* $\mathbf{S}^k K_0 \to K^*$ *a.s.*

Moreover, up to probability distribution, K^* *is the unique random compact set which satisfies* $\mathbf{S}$.

However, using contraction method in probabilistic metric spaces,instead of (6) we can give weaker conditions for the existence and uniqueness of invariant sets.

Theorem 4.2. *Let $\mathcal{E}$ be the set of nonempty random compact sets $A \subset R^n$, and let $\mathbf{S}$ be a random scaling law with $\lambda = ess\sup_{\omega}\ r^{\omega} < 1$. Suppose there exists $Z \in \mathcal{E}$ and a positive number γ such that*

$$P(\{\omega \in \Omega|\ d_{\mathcal{H}}(Z(\omega), \mathbf{S}(Z(\omega))) \geq t\}) \leq \frac{\gamma}{t} \ \textit{for all}\, t > 0. \tag{9}$$

Then there exists $K^* \in \mathcal{E}$ such that $\mathbf{S}(K^*) = K^*$.

Moreover, K^* is unique up to probability distribution.

Proof. Define $f : \mathcal{E} \to \mathcal{E},\ f(A) = \mathbf{S}A$. For $A, B \in \mathcal{E},\ A^i \overset{d}{=} A, B^i \overset{d}{=} B, i \in \{1, ..., N\}$, one checks that

$$F_{f(A),f(B)}(t) = P(\{\omega \in \Omega|\ d_{\mathcal{H}}(f(A), f(B)) < t\}) =$$

$$= P(\{\omega \in \Omega|\ d_{\mathcal{H}}(\cup_{i=1}^{N} S_i(\omega)(A^i(\omega)), \cup_{i=1}^{N} S_i(\omega)(B^i(\omega))) < t\}) \geq$$

$$\geq P(\{\omega \in \Omega|\ \lambda \cdot \max_i \cdot d_{\mathcal{H}}(A^i(\omega)), B^i(\omega)) < t\}) =$$

$$= P(\{\omega \in \Omega|\ \lambda \cdot d_{\mathcal{H}}(A(\omega), B(\omega)) < t\}) = F_{A,B}(\frac{t}{\lambda}) \ \text{ for all } t > 0.$$

It follows that f is a contraction with ratio λ and we can apply the Corollary 3.1 for $r = \lambda$. For the uniqueness, let $\mathcal{C}$ the set of probability distribution of members of $\mathbf{C}$, i.e.

$$\mathcal{C} = \{distA | A \in \mathbf{C}\}.$$

We define on $\mathcal{C}$ the probability metric by

$$F_{\mathcal{A},\mathcal{B}}(t) = \sup_{s<t} \sup\{F_{A,B}(s) | A \overset{d}{=} \mathcal{A},\ B \overset{d}{=} \mathcal{B}\}.$$

It is easy to verify that $\mathcal{S}$ is a contraction map:

$$F_{\mathcal{S}A,\mathcal{S}B}(t) \geq F_{\mathcal{A},\mathcal{B}}(\frac{t}{\lambda}) \text{ for all } t > 0.$$

Let $\mathcal{K}^*$ and $\mathcal{K}^{**}$ such that

$$\mathcal{S}K^* = K^* \ and\ \mathcal{S}K^{**} = K^{**}.$$

As in the proof of the Theorem 3, one can show that

$$F_{\mathcal{K}^*,K^{**}}(t) = 1 \text{ for all } t > 0. \qquad \Box$$

Remark. If condition (6) is satisfied, then (9) holds.

References

[1] M.Arbeiter: Random recursive construction of self-similar fractal measures. The non-compact case. Probab. Theory Related Fields 88(1991), 497-520.
[2] Gh.Constantin, I.Istraţescu: Elements of Probabilistic Analysis, Kluwer Academic Publishers, 1989.
[3] R.Egbert: Product and Quotients of Probabilistic Metric Spaces, Pacific Journal of Mathematics, 24 (1968), no.3, 437-455.
[4] K.J.Falconer: Random fractals, Math. Proc.Cambridge Philos. Soc. 100 (1986), 559-582.
[5] S.Graf: Statistically self-similar fractals, Probab. Theory Related Fields, 74 (1987), 357-392.
[6] J.E.Hutchinson: Fractals and Self Similarity, Indiana University Mathematics Journal, 30 (1981), no.5, 713-747.
[7] J.E.Hutchinson, L.Rüschendorf: Random Fractal Measures via the Contraction Method, Indiana Univ. Math. Jouranal, 47(2), (1998), 471-489.
[8] J.E.Hutchinson, L.Rüschendorf: Random Fractal and probability metrics, Research Report MRR48, (1998) Australian National University.
[9] J.E.Hutchinson, L.Rüschendorf: Selfsimilar Fractals and Selfsimilar Random Fractals,Progress in Probability, 46, (2000), 109-123.
[10] J.Kolumbán, A. Soós: Invariant sets in Menger spaces, Studia Univ. "Babes-Bolyai", Mathematica, 43, 2 (1998), 39-48.
[11] K.Menger: Statistical Metrics, Proc.Nat. Acad. of Sci.,U.S.A. 28 (1942), 535-537.
[12] L.Olsen: Random geometrically graph directed self-similar multifractals, Longman Scientific and Technical, Harlow, 1994.
[13] B.Schweizer, A.Sklar: Statistical Mertic Spaces, Pacific Journal of Mathematics, 10 (1960), no. 1, 313-334.
[14] B.Schweizer, A.Sklar: Probabilistic Mertic Spaces, North Holland, New-York, Amsterdam, Oxford, 1983.
[15] B.Schweizer, A.Sklar, E.Thorp: The Metrization of Statistical Metric Spaces, Pacific Journal of Mathematics, 10 (1960), no. 2, 673-675.
[16] V.M.Sehgal: Some Fixed Point Theorems in Functional Analysis and Probability, Ph.D.Thesis, Wayne State University, 1966.
[17] V.M.Sehgal: A Fixed Point Theorem for Mappings with a Contractive Iterate, Proc. Amer. Math. Soc.,23 (1969), 631-634.
[18] V.M.Sehgal, A.T. Bharucha-Reid: Fixed Points of Contraction Mappings on Probabilistic Metric Spaces, Math. Systems Theory, 6 (1972), 92-102.
[19] H.Sherwood: Complete Probabilistic Metric Spaces, Z.Wahrsch.verw. Geb., 20 (1971), 117-128.
[20] H.Sherwood: E-spaces and their relation to other classes of probabilistic metric spaces, J.London Math. Soc., 44 (1969), 441-448
[21] B.Z.Vulikh: A Brief Course in the Theory of Functions of Real Variable, Mir,1976.

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