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A NOTE ON τ-QUASI-INJECTIVE MODULES

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Abstract. Let τ be a hereditary torsion theory. We mention a characterization of τ-quasi-injective modules, as fully invariant submodules of their τ-injective hull, and we give some properties for such modules. Moreover, the paper studies when τ-quasi-injective modules are quasi-injective or not, in the case of the hereditary torsion theory τ_D whose τ_D-torsion class consists of all semiartinian modules and τ_D-torsionfree class consists of all modules with zero socle.

1. **Preliminaries**

Throughout this paper we will denote by R an associative ring with non-zero identity and by τ a hereditary torsion theory on the category R-mod of left R-modules. All modules considered in the paper will be left unital R-modules.

A module A is said to be semiartinian if every non-zero homomorphic image of A contains a simple submodule [6, Chapter I, Definition 11.4.6]. Let A be a module and let B be a submodule of A. Then A is semiartinian if and only if B and A/B are semiartinian [6, Chapter I, Proposition 11.4.8].

A submodule B of a module A is said to be τ-dense (τ-closed) in A if A/B is τ-torsion (τ-torsionfree). A non-zero module A is called τ-cocritical if A is τ-torsionfree and each of its non-zero submodules is τ-dense in A.

A module A is said to be τ-injective if $Ext^1_R(B, A) = 0$ for every τ-torsion module B. A module A is τ-injective if and only if A is a τ-closed submodule of its injective hull [5, Proposition 8.2]. The class of τ-injective modules is closed under taking direct products, direct summands and extensions [5, Proposition 8.4]. For any module A, we will denote by $E(A)$ and $E_\tau(A)$ the injective hull and the τ-injective hull of A respectively.

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In this paper, a non-zero module which is the τ-injective hull of each of its non-zero submodules will be called minimal τ-injective.

For additional information on torsion theories we refer to [5].

2. **Some properties**

A module A is said to be τ-quasi-injective if whenever B is a τ-dense submodule of A, any $g \in Hom_R(B, A)$ can be extended to $h \in End_R(A)$ [1, Definition 4.1.19].

Remarks. a) Every quasi-injective module is τ-quasi-injective.

b) Every τ-injective module is τ-quasi-injective.

c) A ring R is a τ-quasi-injective R-module if and only if it is τ-injective.

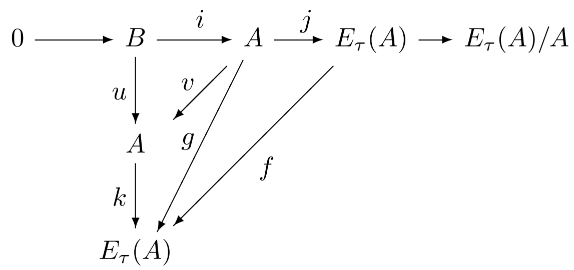
d) If A is a τ-torsion τ-quasi-injective module, then A is quasi-injective.

The following theorem gives a characterization of τ-quasi-injective modules similar to the well known characterization of quasi-injective modules, which are fully invariant submodules of their injective hulls.

Theorem 2.1. *Let A be a module. Then A is τ-quasi-injective if and only if A is a fully invariant submodule of $E_\tau(A)$.*

Proof. We may suppose that $A \neq 0$. Denote $K = End_R(E_\tau(A))$.

Assume first that A is τ-quasi-injective and let $f \in K$. Denote $g = f|_A$ and $B = g^{-1}(A)$. Consider the following commutative diagram



where i, j, k are inclusion monomorphisms and $u : B \to A$ is defined by $u(b) = g(b)$ for every $b \in B$.

We will show that B is a τ-dense submodule of A. The homomorphism g induces a monomorphism $w : A/B \to E_\tau(A)/A$, defined by $w(a + B) = g(a) + A$ for every $a \in A$. Then A/B is τ-torsion because $E_m(A)/A$ is τ-torsion. Hence B is a τ-dense submodule of A.

Since A is τ-quasi-injective, there exists $v \in End_R(A)$ such that $vi = u$. By τ-injectivity of $E_\tau(A)$, there exists $h \in K$ such that $hj = kv$. Thus $h(A) \subseteq A$. Assume $(h-f)(A) \neq 0$. Then $(h-f)(A) \cap A \neq 0$ and there exist $x, y \in A$, $y \neq 0$ such that $y = (h-f)(x)$. It follows that $(h-f)(x) = v(x) - f(x) = y$, hence $f(x) = v(x) - y \in A$. Then $x \in B$ and $y = v(x) - f(x) = 0$, contradiction. Therefore, $(h-f)(A) = 0$, i.e. $f(A) = h(A) \subseteq A$. Hence A is a fully invariant submodule of $E_\tau(A)$.

Suppose now that A is a fully invariant submodule of $E_\tau(A)$. Let B be a τ-dense submodule of A and let $g \in Hom_R(B, A)$. The module $E_\tau(A)/B$ is τ-torsion because $E_\tau(A)/A$ and A/B are τ-torsion. Then g extends to $h \in K$ because $E_\tau(A)$ is τ-injective. Since $h(A) \subseteq A$, g extends to an endomorphism of A. Therefore A is τ-quasi-injective. □

Corollary 2.2. *If every τ-injective module is injective, then every τ-quasi-injective module is quasi-injective.*

Proof. By Theorem 2.1, if A is a τ-quasi-injective module, then A is a fully invariant submodule of $E_\tau(A)$. But $E_\tau(A) = E(A)$. Hence A is a fully invariant submodule of $E(A)$, i.e. A is quasi-injective. □

Remark. By Theorem 2.1 and in a similar way as for quasi-injective modules, it can be easily shown that the class of τ-quasi-injective modules is closed under taking direct summands and any finite direct sum of copies of a τ-quasi-injective module is τ-quasi-injective.

Theorem 2.3. *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of modules and let $h : B \to A \oplus D$ be a monomorphism, where D is a module. If $(hf)(A)$ is a τ-dense submodule of $A \oplus D$ and $A \oplus D$ is τ-quasi-injective, then the above sequence splits.

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle h} & & & & \\ & & A\oplus D & \xleftarrow{\theta} & A\oplus D & & & & \end{array}$$

where $\alpha : A \to A \oplus D$ is the canonical injection. Since $(A \oplus D)/(hf)(A)$ is τ-torsion and $A \oplus D$ is τ-quasi-injective, there exists an endomorphism $\theta : A \oplus D \to A \oplus D$ such that $\theta hf = \alpha$. Let $p : A \oplus D \to A$ be the canonical projection and define $\gamma : B \to A$ by $\gamma = p\theta h$. Then $\gamma f = p\theta hf = p\alpha = 1_A$, hence the above sequence splits. □

Corollary 2.4. *Let $f : A \to B$ be a monomorphism of modules. If B is τ-torsion and $A \oplus B$ is τ-quasi-injective, then $A \oplus B$ is τ-injective if and only if B is τ-injective.*

Proof. The "if" part is obvious.

For the "only if" part, in the Theorem 2.3, let $h : B \to A \oplus B$ be the canonical injection. Since B is τ-torsion, A and $B/f(A)$ are τ-torsion. Hence $(A \oplus B)/(hf)(A) \cong (A \oplus B)/f(A)$ is τ-torsion. By Theorem 2.3, $f(A)$ is a direct summand of B, hence A is τ-injective. Therefore $A \oplus B$ is τ-injective. □

3. **The Dickson torsion theory**

In this section we will establish further results in the case of a particular hereditary torsion theory, namely the Dickson torsion theory.

For let $\mathcal{T}$ be the class of all semiartinian R-modules and let $\mathcal{F}$ be the class of all R-modules with zero socle. Then $\tau_D = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory. The corresponding Gabriel filter F consists of all τ_D-dense left ideals of R (i.e. all left ideals of R with R/I left semiartinian as an R-module).

An R-module D is τ_D-injective if any homomorphism from any left ideal $I \in F$ to D extends to R or equivalently if D is injective with respect to every short exact sequence of modules $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, where C is τ_D-torsion (i.e. C is semiartinian).

We consider now the following generalization of injectivity for modules. An R-module D is said to be m-injective if for every maximal left ideal M of R the R-module D is injective with respect to the inclusion monomorphism $u : M \to R$ [2, Definition 1].

The notions of τ_D-injectivity and m-injectivity are in fact the same [2, Theorem 6]. By this reason, in the sequel we will use the notation m instead of τ_D. For instance, injective and quasi-injective modules with respect to the Dickson torsion theory will be called m-injective and m-quasi-injective modules respectively.

From the general context of torsion theories it follows that every module A has an m-injective hull, denoted by $E_m(A)$, contained in $E(A)$, unique up to an isomorphism.

We have seen that every quasi-injective module is τ-quasi-injective. For the Dickson torsion theory we will give several cases when quasi-injectivity and m-quasi-injectivity are or are not the same.

Proposition 3.1. *Let R be either left semiartinian or left m-cocritical. Then every m-quasi-injective R-module is quasi-injective.*

Proof. In both cases every, every non-zero left ideal is m-dense in R, hence every m-injective module is injective. Now the result follows by Corollary 2.2. □

Corollary 3.2. *Let R be a commutative noetherian domain with dim $R \leq 1$. Then every m-quasi-injective R-module is quasi-injective.*

Proof. By hypotheses, every m-injective module is injective [2, Corollary 13]. Now the result follows by Corollary 2.2. □

In the sequel we will see that there exist m-quasi-injective modules which are not m-injective and even quasi-injective modules which are not m-injective.

Theorem 3.3. *Let A be an m-quasi-injective module which is not m-injective and denote $M = E_m(A)$. Consider the Loewy series of M/A*

$$0 = S_0(M/A) \subseteq S_1(M/A) \subseteq \cdots \subseteq S_\alpha(M/A) \subseteq S_{\alpha+1}(M/A) \subseteq \ldots$$

where, for each ordinal $\alpha \geq 0$,

$$S_{\alpha+1}(M/A)/S_\alpha(M/A) = Soc((M/A)/S_\alpha(M/A))$$

and if α is a limit ordinal, then

$$S_\alpha(M/A) = \bigcup_{0 \leq \beta < \alpha} S_\beta(M/A)\,.$$

For every ordinal $\alpha \geq 0$, let M_α be a submodule of M be such that $S_\alpha(M/A) = M_\alpha/A$.

Then every non-zero proper submodule M_α of M is m-quasi-injective, but not m-injective.

Proof. Let $\alpha \geq 1$ be an ordinal such that M_α is a proper submodule of M and let $f \in End_R(M)$. Since A is m-quasi-injective, $f(A) \subseteq A$ by Theorem 2.1. Then f induces an endomorphism $f^* \in End_R(M/A)$. Since $M_\alpha/A = S_\alpha(M/A)$ is fully invariant [4, 3.11, p.25], $f^*(M_\alpha/A) \subseteq M_\alpha/A$, therefore $f(M_\alpha) \subseteq M_\alpha$, i.e. M_α is m-quasi-injective. On the other hand, M_α is a proper submodule of $E_m(A) = M$, hence M_α is not m-injective. □

Theorem 3.4. *Let S be a simple module which is not m-injective and denote $M = E_m(S)$. Consider the Loewy series of M*

$$0 = S_0(M) \subseteq S_1(M) \subseteq \cdots \subseteq S_\alpha(M) \subseteq S_{\alpha+1}(M) \subseteq \dots$$

where, for each ordinal $\alpha \geq 0$, $S_{\alpha+1}(M)/S_\alpha(M) = Soc(M/S_\alpha(M))$ and if α is a limit ordinal, then $S_\alpha(M) = \bigcup_{0 \leq \beta < \alpha} S_\beta(M)$.

Then every non-zero proper submodule $S_\alpha(M)$ of M is quasi-injective, but not m-injective.

Proof. Let $\alpha \geq 1$ be an ordinal such that $S_\alpha(M)$ is a proper submodule of M. Then $S_\alpha(M)$ is a fully invariant submodule of M [4, 3.11, p.25], therefore m-quasi-injective by Theorem 2.1. Also $S_\alpha(M)$ is semiartinian as a submodule of the semiartinian module M. It follows that $S_\alpha(M)$ is quasi-injective. Since $M = E_m(S)$ is minimal m-injective, $S_\alpha(M)$ is not m-injective. □

We have noted that every quasi-injective module is m-quasi-injective. The converse is not true, as we can see in the following example.

Example 3.5. Let R be a unique factorization domain such that every maximal ideal of R is not principal. Then R is an m-injective R-module which is not injective [2, Theorem 15]. Hence R is m-quasi-injective. Since R is quasi-injective if and only if R is injective, it follows that R is not quasi-injective.

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