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# NORM ESTIMATES, COEFFICIENT ESTIMATES AND SOME PROPERTIES OF SPIRAL-LIKE FUNCTIONS

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*Dedicated to Professor Petru T. Mocanu on his $70^{th}$ birthday*

**Abstract**. This is a survey of the author's talk at the VIIIth Romanian-Finnish Seminar in Iassy, Romania, in 23-27 August 1999. We shall state the sharp estimates of the norms of pre-Schwarzian and Schwarzian derivatives of spiral-like functions and about the optimal growth estimates of coefficients of them. We shall also remark that some spiral-like function $f(z) = z + a_2 z^2 + \cdots$ is normalized and univalent on the unit disk $\mathbb{D}$ but satisfies $a_2 f(z) + 1 = 0$ for some $z \in \mathbb{D}$.

## 1. Introduction

We consider an analytic function $f$ on the unit disk $\mathbb{D}$ normalized so that $f(0) = f'(0) - 1 = 0$. For a constant $\beta \in (-\pi/2, \pi/2)$, such a function $f$ is called *$\beta$-spiral-like* if $f$ is univalent on $\mathbb{D}$ and for any $z \in \mathbb{D}$, the $\beta$-logarithmic spiral $\{f(z)\exp(-e^{i\beta}t); t \geq 0\}$ is contained in $f(\mathbb{D})$. It is equivalent to the analytic condition that $\Re(e^{-i\beta} z f'(z)/f(z)) > 0$ in $\mathbb{D}$. We denote by $SP(\beta)$ the set of $\beta$-spiral-like functions. We call $f_\beta(z) := z(1-z)^{-2e^{i\beta}\cos\beta} \in SP(\beta)$ the $\beta$-spiral Koebe function. Note that $SP(0)$ is the set of starlike functions and that $f_0(z) = z(1-z)^{-2}$ is the Koebe function. The $\beta$-spiral Koebe function conformally maps the unit disk onto the complement of the $\beta$-logarithmic spiral $\{f_\beta(-e^{-2i\beta})\exp(-e^{i\beta}t); t \leq 0\}$ in $\mathbb{C}$. For the known results about these classes of the functions, see, for example, [1].

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## 2. **Norm estimates**

For a locally univalent holomorphic function $f$, we define

$$T_f = \frac{f''}{f'} \quad \text{and} \quad S_f = (T_f)' - \frac{1}{2}(T_f)^2,$$

which are said to be the *pre-Schwarzian derivative* (or nonlinearity) and the *Schwarzian derivative* of $f$, respectively. For a locally univalent function $f$ in $\mathbb{D}$, we define the norms of $T_f$ and $S_f$ by

$$\|T_f\|_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)|T_f(z)| \quad \text{and} \quad \|S_f\|_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^2|S_f(z)|,$$

respectively.

As well as $\|S_f\|_2$, the norm $\|T_f\|_1$ has a significant meaning in the theory of Teichmüller spaces. For example, see [8], [2] and [13].

We shall give the best possible estimate of the norms of pre-Schwarzian derivatives for the class $SP(\beta)$.

**Main Theorem 1** ([9]). *For any $f \in SP(\beta)$, where $\beta \in (-\pi/2, \pi/2)$, we have the following.*

I) *In the case $|\beta| \leq \pi/3$, we have*

$$\|T_f\|_1 \leq \|T_{f_\beta}\|_1 = 2|2+e^{2i\beta}|. \tag{1}$$

II) *In the case $|\beta| > \pi/3$, we have $\|T_f\|_1 \leq \|T_{f_\beta}\|_1$, where*

$$\|T_{f_\beta}\|_1 = \max_{0\leq m\leq \frac{4}{3}\sin|\beta|} 2m\cos\beta\left(1+\sqrt{\frac{m^2+4-4m\sin|\beta|}{m^2+1-2m\sin|\beta|}}\right) \quad \textit{and} \tag{2}$$

$$2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1 < 2\left(1+\frac{4}{3}\sin 2|\beta|\right). \tag{3}$$

*In particular, $\|T_{f_\beta}\|_1 \to 2$ as $|\beta| \to \pi/2$.*

*In both cases, the equality $\|T_f\|_1 = \|T_{f_\beta}\|_1$ holds if and only if $f$ is a rotation of the $\beta$-spiral Koebe function, i.e., $f(z) = (1/\varepsilon)f_\beta(\varepsilon z)$ for some $|\varepsilon| = 1$.*

The proof of Main Theorem 1 is in [9]. From the proof, if $|\beta| \leq \pi/3$, the function $(1-|z|^2)|T_{f_\beta}(z)|$ does not attain its supremum in $\mathbb{D}$. However if $|\beta| > \pi/3$, it does since

$$\max_{\partial\mathbb{D}\ni z_0} \limsup_{\mathbb{D}\ni z\to z_0}(1-|z|^2)|T_{f_\beta}(z)| = 2|2+e^{2i\beta}| < \|T_{f_\beta}\|_1.$$

This phenomenon of *phase transition* seems to be quite interesting.

**Remark.** Clearly, the $\beta$-spiral Koebe function $f_\beta$ converges to $id_{\mathbb{D}}$ (which is bounded) locally uniformly on $\mathbb{D}$ as $|\beta| \to \pi/2$ but does not converge to it with respect to the norm $\|\cdot\|_1$ since $\lim_{|\beta|\to\pi/2} \|T_{f_\beta}\|_1 = 2$. On the other hand, it is known that a normalized analytic function $f$ is bounded if $\|T_f\|_1 < 2$. In fact, the value 2 is the least one of the norms of unbounded normalized analytic functions.

We would also like to mention the related works about norm estimates of pre-Schwarzian derivatives in other classes by Shinji Yamashita [11] and Toshiyuki Sugawa [10].

**Theorem 2.1.** *Let $0 \le \alpha < 1$ and $f$ be a normalized analytic function.*

*If $f$ is starlike of order $\alpha$, i.e., $\Re(zf'(z)/f(z)) > \alpha$, then $\|T_f\|_1 \le 6 - 4\alpha$.*

*If $f$ is convex of order $\alpha$, i.e., $\Re(1+zf''(z)/f'(z)) > \alpha$, then $\|T_f\|_1 \le 4(1-\alpha)$.*

*If $f$ is strongly starlike of order $\alpha$, i.e., $\arg(zf'(z)/f(z)) < \pi\alpha/2$, then $\|T_f\|_1 \le M(\alpha)+2\alpha$, where $M(\alpha)$ is a specified constant depending only on $\alpha$ satisfying $2\alpha < M(\alpha) < 2\alpha(1+\alpha)$.*

*All of the bounds are sharp.*

On the other hand, we also obtain the estimate of the norms of Schwarzian derivatives of $\beta$-spiral-like functions.

**Main Theorem 2** ([9]). *Assume $|\beta| < \pi/2$. For any $f \in SP(\beta)$, $\|S_f\|_2 \le \|S_{f_\beta}\|_2 = 6$.*

In the rest of this article, we shall state two remarks about spiral-like functions.

## 3. Order estimates of the coefficients

Knowing the norm $\|T_f\|_1$ enables us to estimate the growth of coefficients of $f$. For example, the following holds.

**Theorem 3.1** (cf.[7]). *Let $(3/2) < \lambda \le 3$. For a normalized analytic function $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ such that $\|T_f\|_1 \le 2\lambda$, it holds that $a_n = O(n^{\lambda-2})$ as $n \to +\infty$. This order estimate is best possible.*

However the sharp estimate of coefficients of $f \in SP(\beta)$ has been already obtained by Zamorski [12] in 1960. We would like to remark that we can derive the sharp growth estimate of coefficients of $f \in SP(\beta)$ from this.

**Theorem 3.2** (Zamorski). *If $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is in $SP(\beta)$ and $|\beta| < \pi/2$, then*

$$|a_n| \leq \prod_{k=1}^{n-1} \left|1 + \frac{e^{2i\beta}}{k}\right| \tag{4}$$

*for $n \geq 2$. The equality in (4) holds for some $n \geq 2$ if and only if $f$ is a rotation of the $\beta$-spiral Koebe function $f_\beta$.*

**Remark.** This is also shown in terms of generalized spiral-like functions by C. Burniak, J. Stankiewicz and Z. Stankiewicz [4](1980).

**Corollary 3.1.** *Let $|\beta| < \pi/2$ and $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be a $\beta$-spiral-like function. Then it holds that*

$$a_n = O(n^{\cos 2\beta}) \quad (n \to +\infty). \tag{5}$$

*This order estimate is sharp.*

**Remark.** In the case $|\beta| < \pi/4$, this is shown by Basgöze and Keogh in [3](1970).

## 4. **Strongly normalized univalent functions are not always holomorphic.**

The following is known.

**Theorem 4.1.** *For a holomorphic function $\phi$ on a simply connected domain $A$, there exists a locally univalent meromorphic function $f$ on $A$ such that*

$$S_f = \phi.$$

*The solution is unique up to postcomposition of an arbitrary Möbius transformation.*

We assume $A = \mathbb{D}$. Nehari showed that if $\|\phi\|_2 = \sup_{z\in\mathbb{D}} |\phi(z)|(1-|z|^2)^2 \leq 2$, then $f$ is univalent and meromorphic on $\mathbb{D}$. It is well-known that if $f$ is *strongly normalized*, i.e., $f(0) = f'(0) - 1 = f''(0) = 0$, then $f$ is holomorphic on $\mathbb{D}$. Since for a normalized analytic function $f(z) = z + a_2z^2 + \cdots$, $g := f/(a_2f + 1)$ is strongly normalized and $\|S_f\|_2 = \|S_g\|_2$, we have the following.

**Proposition 4.1** ([6], [5] Corollary 2). *If a normalized analytic function* $f(z) = z + a_2 z^2 + \cdots$ *satisfies* $\|S_f\|_2 \leq 2$*, then* $f$ *is univalent and* $a_2 f + 1 \neq 0$ *on* $\mathbb{D}$.

In [5] Chuaqui and Osgood remark that a strongly normalized univalent function $f$ is not always holomorphic if $\|S_f\|_2 > 2$. Spiral-like functions are examples for this fact.

**Theorem 4.2.** *If* $|\beta|$ *is sufficiently close to* $\pi/2$*, the* $\beta$*-spiral-Koebe function* $f_\beta(z) = z + a_2 z^2 + \cdots$ *satisfies* $a_2 f_\beta(z) + 1 = 0$ *for some* $z \in \mathbb{D}$.

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