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ON A CLASS OF CERTAIN ANALYTIC FUNCTIONS OF COMPLEX ORDER

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Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. We introduce a class, namely, $H_n(b, M)$ of certain analytic functions. For this class we determine sufficient condition in terms of coefficients, coefficient estimate, maximization theorem concerning the coefficients, and radius problem.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. We use Ω to denote the class of functions $w(z)$ in U satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. For a function $f(z)$ in A, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N = \{1, 2, \dots, \}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [11]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to A is in the

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class $F_n(b, M)$ if and only if

$$\left| \frac{b - 1 + \dfrac{D^{n+1} f(z)}{D^n f(z)}}{b} - M \right| < M, \quad z \in U, \tag{1.5}$$

where $M > \frac{1}{2}$ and $b \neq 0$, complex.

It follows by Kulshrestha [6] that $g(z) \in H_0\ (1, M) = F(1, M)$ if and only if for $z \in U$

$$\frac{zg'(z)}{g(z)} = \frac{1 + w(z)}{1 - mw(z)}, \tag{1.6}$$

where $m = 1 - \frac{1}{M}\ \left(M > \frac{1}{2}\right)$ and $w(z) \in \Omega$.

One can easily show that $f(z) \in H_n(b, M)$ if and only if there is a function $g(z) \in H_0(1, M) = F(1, M)$ such that

$$D^n f(z) = z \left[\frac{g(z)}{z}\right]^b. \tag{1.7}$$

Thus from (1.6) and (1.7) it follows that $f(z) \in H_n(b, M)$ if and only if for $z \in U$

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + [b(1 + m) - m]w(z)}{1 - mw(z)}, \tag{1.8}$$

where $m = 1 - \frac{1}{M}\ \left(M > \frac{1}{2}\right)$ and $w(z) \in \Omega$.

By giving specific values to n, b and M, we obtain the following important subclasses studied by various authors in earlier works:

(1) $H_0(b, M) = F(b, M)$ (Nasr and Aouf [7]) and $H_1(b, M) = G(b, M)$ (Nasr and Aouf [8]).

(2) $H_0(\cos\lambda e^{-i\lambda}, M) = F_{\lambda,M}$ and $H_1(\cos\lambda e^{-i\lambda}, M) = G_{\lambda,M}$ $\left(|\lambda| < \frac{\pi}{2}\right)$ (Kulshrestha [4]).

(3) $H_0((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = S^\lambda(\alpha)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1\right)$ (Libera [6]) and $H_1((1-\alpha)\cos\lambda e^{-i\lambda}, \infty) = C^\lambda(\alpha)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1\right)$ (Chichra [3] and Sizuk [14]).

(4) $H_0(b, M) = S(1 - b)$ (Nasr and Aouf [9]) and $H_1(b, M) = C(b)$ (Wiatrowski [15] and Nasr and Aouf [10]).

(5) $H_0((1 - \alpha)\cos\lambda e^{-i\lambda}, M) = F_M(\lambda, \alpha)$ and $H_1((1 - \alpha)\cos\lambda e^{-i\lambda}, M) = G_M(\lambda, \alpha)$ $\left(|\lambda| < \frac{\pi}{2},\ 0 \leq \alpha < 1\right)$ (Aouf [1,2]).

(6) $H_0(1,1) = F(1,1)$ Singh [12] and $H_0(1,M) = F(1,M)$ (Singh and Singh [13]).

From the definitions of the classes $F(b,M)$ and $H_n(b,M)$, we observe that

$$f(z) \in H_n(b,M) \text{ if and only if } D^n f(z) \in F(b,M). \tag{1.9}$$

The purpose of the present paper is to determine sufficient condition in terms of coefficients for function belong to $H_n(b,M)$, coefficient estimate, and maximization of $|a_3 - \mu a_2^2|$ on the class $H_n(b,M)$ for complex value of μ. Further we obtain the radius of disc in which $\operatorname{Re}\left\{\dfrac{D^{n+1}f(z)}{D^n f(z)}\right\} > 0$, wherever $f(z)$ belongs to $H_n(b,M)$.

2. A sufficient condition for a function to be in $H_n(b,M)$

Theorem 1. *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{k=2}^{\infty}\{(k-1) + |b(1+m) + m(k-1)|\}k^n|a_k| \le |b(1+m)|, \tag{2.1}$$

holds, then $f(z)$ belongs to $H_n(b,M)$, where $m = 1 - \dfrac{1}{M}$ $\left(M > \dfrac{1}{2}\right)$.

Proof. Suppose that the inequality (2.1) holds. Then we have for $z \in U$

$$|D^{n+1}f(z) - D^n f(z)| - |b(1+m)D^n f(z) + m(D^{n+1}f(z) - D^n f(z))|$$

$$= \left|\sum_{k=2}^{\infty} k^n(k-1)a_k z^k\right| - \left|b(1+m)\left\{z + \sum_{k=2}^{\infty} k^n a_k z^k\right\} + m\sum_{k=2}^{\infty} k^n(k-1)a_k z^k\right| \le$$

$$\le \sum_{k=2}^{\infty} k^n(k-1)|a_k|r^k - \left\{|b(1+m)|r - \sum_{k=2}^{\infty}|b(1+m) + m(k-1)|k^n|a_k|r^k\right\} =$$

$$= \sum_{k=2}^{\infty}\{(k-1) + |b(1+m) + m(k-1)|\}k^n|a_k|r^k - |b(1+m)|r.$$

Letting $r \to {}^{-}1$, then we have

$$|D^{n+1}f(z) - D^n f(z)| - |b(1+m)D^n f(z) + m(D^{n+1}f(z) - D^n f(z))| =$$

$$= \sum_{k=2}^{\infty}\{(k-1) + |b(1+m) + m(k-1)|\}k^n|a_k| - |b(1+m)| \le 0, \text{ by (2.1)}.$$

Hence it follows that

$$\left|\frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{b(1+m) + m\left\{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right\}}\right| < 1, \quad z \in U.$$

Letting

$$w(z) = \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{b(1+m) + m\left\{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1\right\}},$$

then $w(0) = 0$, $w(z)$ is analytic in $|z| < 1$ and $|w(z)| < 1$. Hence we have

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1 + [b(1+m) - m]w(z)}{1 - mw(z)}$$

which shows that $f(z)$ belongs to $H_n(b, M)$.

3. **Coefficient estimate**

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $H_n(b, M)$, $z \in U$.*

(a) If $2m(k-1)\mathrm{Re}\,\{b\} > (k-1)^2(1-m) - |b|^2(1+m)$, let

$$N = \left[\frac{2m(k-1)\mathrm{Re}\,\{b\}}{(k-1)^2(1-m) - |b|^2(1+m)}\right], \quad k = 2, 3, \ldots, j-1.$$

Then

$$|a_j| \leq \frac{1}{j^n(j-1)!}\prod_{k=2}^{j}|b(1+m) + (k-2)m|, \tag{3.1}$$

for $j = 2, 3, \ldots, N+2$; and

$$|a_j| \leq \frac{1}{j^n(j-1)(N+1)!}\prod_{k=2}^{N+3}|b(1+m) + (k-2)m|, \quad j > N+2. \tag{3.2}$$

(b) If $2m(k-1)\mathrm{Re}\,\{b\} \leq (k-1)^2(1-m) - |b|^2(1+m)$, then

$$|a_j| \leq \frac{(1+m)|b|}{j^n(j-1)}, \ \text{for } j \geq 2, \tag{3.3}$$

where $m = 1 - \dfrac{1}{M}$ $\left(M > \dfrac{1}{2}\right)$ and $b \neq 0$, complex.

The inequalities (3.1) and (3.3) are sharp.

Proof. Since $f(z) \in H_n(b, M)$, so from (1.8) we have that

$$\sum_{k=2}^{\infty} k^n(k-1)a_k z^k = \left\{b(1+m)z + \sum_{k=2}^{\infty} k^n[b(1+m) + m(k-1)]a_k z^k\right\} w(z) \tag{3.4}$$

which is equivalent to

$$\sum_{k=2}^{j} k^n(k-1)a_k z^k + \sum_{k=2}^{\infty} d_k z^k =$$

$$= \left\{ b(1+m)z + \sum_{k=2}^{j-1} k^n [b(1+m) + m(k-1)] \right\} a_k z^k w(z),$$

where d_j's are some complex numbers.

Then since $|w(z)| < 1$, we have

$$\left| \sum_{k=2}^{j} k^n (k-1) a_k z^k + \sum_{j=k+1}^{\infty} d_k z^k \right| \leq$$

$$\leq \left| b(1+m)z + \sum_{k=2}^{j-1} k^n [b(1+m) + m(k-1)] a_k z^k \right|. \tag{3.5}$$

Squaring both sides of (3.5) and integrating round $|z| = r < 1$, we get, after taking the limit when $r \to 1$

$$j^{2n}(j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2 +$$

$$+ \sum_{k=2}^{j-1} k^{2n} \{ |b(1+m) + m(k-1)|^2 - (k-1)^2 \} |a_k|^2. \tag{3.6}$$

Now there may be following two cases:

(a) Let $2m(k-1)\text{Re}\,\{b\} > (k-1)^2(1-m) - (1+m)|b|^2$. Suppose that $j \leq N+2$; then for $j = 2$, (3.7) gives

$$|a_2| \leq \frac{(1+m)|b|}{2^n}$$

which gives (3.1) for $j = 2$. We establish (3.1), by mathematical induction. Suppose (3.1) is valid for $k = 2, 3, \ldots, j-1$. Then it follows from (3.6)

$$j^{2n}(j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2 +$$

$$+ \sum_{k=2}^{j-1} k^{2n} \{ |b(1+m) + m(k-1)|^2 - (k-1)^2 \} \frac{1}{k^{2n}((k-1)!)^2} \prod_{p=2}^{k} |b(1+m) + (p-2)m|^2 =$$

$$= \frac{1}{((j-1)!)^2} \prod_{k=2}^{j} |b(1+m) + (k-2)m|^2.$$

Thus, we get

$$|a_j| \leq \frac{1}{j^n (j-1)!} \prod_{k=2}^{j} |b(1+m) + (k-2)m|,$$

which completes the proof of (3.1).

Next, we suppose $j > N+2$. Then (3.6) gives

$$j^{2n}(j-1)^2 |a_j|^2 \leq (1+m)^2 |b|^2 +$$

$$+\sum_{k=2}^{N+1} k^{2n}\{|b(1+m)+m(k-1)|^2-(k-1)^2\}|a_k|^2+$$

$$+\sum_{k=N+3}^{j-1} k^{2n}\{|b(1+m)+m(k-1)|^2-(k-1)^2\}|a_k|^2 \leq$$

$$\leq (1+m)^2|b|^2+\sum_{k=2}^{N+2}\{|b(1+m)+m(k-1)|^2-(k-1)^2\}|a_k|^2.$$

On substituting upper estimates for $a_2, a_3, \ldots, a_{N+2}$ obtained above, and simplifying, we obtain (3.2).

(b) Let $2m(k-1)\text{Re }\{b\} \leq (k-1)^2(1-m)-(1+m)|b|^2$, then it follows from (2.7)

$$j^{2n}(j-1)^2|a_j|^2 \leq (1+m)^2|b|^2, \quad (j \geq 2)$$

which proves (3.3).

The bounds in (3.1) are sharp for the function $f(z)$ given by

$$D^n f(z) = \begin{cases} \dfrac{z}{(1-mz)^{\frac{b(1+m)}{m}}}, & m \neq 0, \\ z\exp(bz), & m = 0. \end{cases} \tag{3.7}$$

The bounds in (3.3) are sharp for the function $f_k(z)$ given by

$$D^n f_k(z) = \begin{cases} \dfrac{z}{(1-mz^{k-1})\dfrac{b(1+m)}{m(k-1)}}, & m \neq 0, \\ \\ z\exp\left(\dfrac{b}{k-1}z^{k-1}\right), & m = 0. \end{cases} \tag{3.8}$$

4. **Maximization of** $|a_3-\mu a_2^2|$

We shall need in our discussion the following lemma:

Lemma 1. [5] *Let* $w(z)=\sum_{k=1}^{\infty} c_k z^k \in \Omega$, *if* μ *is any complex number, then*

$$|c_2-\mu c_1^2| \leq \max\{1, |\mu|\} \tag{4.1}$$

for any complex μ. *Equality in (4.1) may be attained with the functions* $w(z)=z^2$ *and* $w(z)=z$ *for* $|\mu|<1$ *and* $|\mu| \geq 1$, *respectively.*

Theorem 3. *If a function $f(z)$ defined by (1.1) is in the class $H_n(b, M)$ and μ is any complex number, then*

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} \max\{1, |d|\}, \tag{4.2}$$

where

$$d = \frac{2 \cdot 3^n \mu b(1+m)}{2^{2n}} - [b(1+m) + m]. \tag{4.3}$$

The result is sharp.

Proof. Since $f(z) \in H_n(b, m)$, we have

$$w(z) = \frac{D^{n+1}f(z) - D^n f(z)}{[b(1+m) - m]D^n f(z) + mD^{n+1}f(z)} =$$

$$= \frac{\sum\limits_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{b(1+m) + \sum_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^{k-1}} =$$

$$= \frac{\sum\limits_{k=2}^{\infty} k^n (k-1) a_k z^{k-1}}{b(1+m)} \left[1 + \frac{\sum\limits_{k=2}^{\infty} k^n [b(1+m) + m(k-1)] a_k z^{k-1}}{b(1+m)} \right]^{-1}. \tag{4.4}$$

Now compare the coefficients of z and z^2 on both sides of (4.4). We thus obtain

$$a_2 = \frac{b(1+m)}{2^n} c_1, \tag{4.5}$$

and

$$a_3 = \frac{b(1+m)}{2 \cdot 3^n} \{c_2 + [b(1+m) + m]c_1^2\}. \tag{4.6}$$

Hence

$$a_3 - \mu a_2^2 = \frac{b(1+m)}{2 \cdot 3^n} [c_2 - dc_1^2], \tag{4.7}$$

where

$$d = \frac{2 \cdot 3^n \mu b(1+m)}{2^{2n}} [b(1+m) + m].$$

Taking modulus both sides in (4.7), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} |c_2 - dc_1^2|. \tag{4.8}$$

Using Lemma 1 in (4.8), we have

$$|a_3 - \mu a_2^2| \leq \frac{|b(1+m)|}{2 \cdot 3^n} \max\{1, |d|\}.$$

Finally, the assertion (4.2) of Theorem 3 is sharp in view of the fact that the assertion (4.1) of Lemma 1 is sharp.

5. Radius Theorem

The following theorem may be obtained with the help of (1.9) and Theorem 3 of Nasr and Aouf [7].

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $H_n(b, M)$. Then*

$$\text{Re}\,\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > 0 \text{ for } |z| < r_n,$$

where

$$r_n = 2\left\{|b|(1+m) + \left[|b|^2(1+m)^2 - 4\left\{\text{Re}\,(b)\left(\frac{1+m}{m}\right) - 1\right\} - 1\right]^{\frac{1}{2}}\right\}^{-1} \tag{5.1}$$

such that

$$|b|^2(1+m)^2 \geq 4\left\{\text{Re}\,(b)\left(\frac{1+m}{m}\right) - 1\right\}.$$

The result is sharp for the function $f_0(t)$, where

$$D^n f_0(z) = z(1-mz)^{-b\left(\frac{1+m}{m}\right)} \tag{5.2}$$

and

$$t = \frac{r\left[r - m\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right]}{m\left[1 - mr\left(\frac{\bar{b}}{b}\right)^{\frac{1}{2}}\right]}.$$

Remarks on Theorem 4.

(i) Putting $n = 0$, we get the sharp radius of starlikeness of the class $F(b, M)$ studied by Nasr and Aouf [7].

(ii) Putting $n = 1$, we get the sharp radius of convexity of the class $G(b, M)$ which is investigated by Nasr and Aouf [8].

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