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ON SOME CLASSES OF HOLOMORPHIC FUNCTIONS

DORINA RĂDUCANU

Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. In this note we define two classes of functions, which are called α-starlike and α-harmonic starlike and we obtain some properties concerning these classes.

1. **Introduction and preliminaries**

Let $\mathbb{C}^n$ be the space of n-complex variables $z = (z_1, \ldots, z_n)$ with the norm $\|z\| = \max\limits_{1\le k\le n} |z_k|$. The unit polydisc $\{z \in \mathbb{C}^n : \ \|z\| < 1\}$ is denoted by P.

Let $H(P)$ be the family of all holomorphic functions from P into $\mathbb{C}$. The Fréchet derivative of $f \in H(P)$ is

$$Df(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right), \quad z \in P$$

and $D^2 f(z) = \left(\frac{\partial^2 f}{\partial z_k \partial z_j}(z)\right)_{1\le k,j\le n}$ is the Fréchet derivative of the second order of f.

Let A denote the class of all functions $f \in H(P)$ which satisfy the conditions $f(0) = 0$ and $\frac{\partial f}{\partial z_k}(0) = 1$, $1 \le k \le n$.

In several papers K. Dobrowolska, J. Dziubinski, R. Sitarski [1], [2] and E. Janiec [4] have studied the subclasses of the class A consisting in starlike and convex functions.

Let $S^*(P)$ be the class of all functions $f \in A$, $f(z) \neq 0$ for all $z \in P \setminus \{0\}$, satisfying the condition

$$Re\frac{zDf(z)'}{f(z)} > 0, \quad \text{for} \quad z \in P \tag{1}$$

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where $Df(z)'$ is the transpose of $Df(z)$. The functions of this class are called starlike on P.

Let $S^c(P)$ be the class of all functions $f \in A$, $zDf'(z) \neq 0$, $z \in P \setminus \{0\}$, for which

$$Re\left(1 + \frac{zD^2f(z)z'}{zDf(z)'}\right) > 0, \quad \text{for} \quad z \in P \tag{2}$$

where z' is the transpose of z. The class $S^c(P)$ is the class of convex functions on P.

We shall use the following theorem to prove our results.

Theorem 1. [3] *Let q be a holomorphic and univalent function on $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ without at most one point $\zeta \in \partial U$, which is a simple pole. Let $p : P \to \mathbb{C}$ be a holomorphic function on P with $p(0) = q(0)$. If $p(P) \not\subset q(U)$, then there exist $\zeta_0 \in \partial U$, $r_0 \in (0,1)$, $z_0 \in r_0\overline{P}$ and $m \geq 1$ such that*

$$p(z_0) = q(\zeta_0) \tag{3}$$

$$z_0 Df(z_0)' = m\zeta_0 q'(\zeta_0) \tag{4}$$

$$Re\left(1 + \frac{z_0 D^2 f(z_0) z_0'}{z_0 Df(z_0)'}\right) \geq m\, Re\left(1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right). \tag{5}$$

2. **Main results**

Let α be a complex number. A function $f \in A$, $f(z) \neq 0$, $z \in P \setminus \{0\}$ is called α-starlike on P if the function

$$G(z) = (1 - \alpha)f(z) + \alpha z Df(z)', \quad \text{for} \quad z \in P \tag{6}$$

is a starlike function on P. We denote by $S^*_\alpha(P)$ the class of α-starlike functions on P.

Since $G \in S^*(P)$, from (1) and (6) it follows that a function f is α-starlike on P if

$$Re\left[p(z) + \alpha\frac{zDp(z)'}{1 - \alpha + \alpha p(z)}\right] > 0, \quad \text{for all} \quad z \in P, \tag{7}$$

where $p(z) = \dfrac{zDf(z)'}{f(z)}$.

The definitions of the classes $S^*(P)$, $S^c(P)$ and $S^*_\alpha(P)$ imply immediately $S^*_0(P) = S^*(P)$ and $S^*_1(P) = S^c(P)$.

Theorem 2. *If $f \in S^*_\alpha(P)$ and $\alpha \in \mathbb{C}$ with $\left|\alpha - \dfrac{1}{2}\right| \leq \dfrac{1}{2}$, then $f \in S^*(P)$.*

Proof. We assume that $Re\dfrac{zDf(z)'}{f(z)} \not> 0$ for some $z \in P$. Let $q : \overline{U} \setminus \{1\} \to \mathbb{C}$ be the function defined by $q(z) = \dfrac{1+z}{1-z}$.

If $p(z) = \dfrac{zDf(z)'}{f(z)}$, $z \in P$ then we have $p(0) = q(0) = 1$ and $p(P) \not\subset q(U)$. From Theorem 1 there exist $\xi_0 \in \partial U$, $r_0 \in (0,1)$ and $z_0 \in r_0\overline{P}$ such that $p(z_0) = q(\zeta_0)$ and $z_0Dp(z_0)' = m\zeta_0 q'(\zeta_0)$, $m \geq 1$. It follows $Re\, p(z_0) = Re\, q(\zeta_0) = 0$ and $z_0Dp(z_0)' < 0$. We obtain

$$Re\left[p(z_0) + \alpha\frac{z_0Dp(z_0)'}{1-\alpha+\alpha p(z_0)}\right] = \frac{z_0Dp(z_0)'}{|1-\alpha+\alpha p(z_0)|^2}Re(\alpha - |\alpha|^2).$$

Since $\left|\alpha - \dfrac{1}{2}\right| \leq \frac{1}{2}$ it follows $Re\left[p(z_0) + \dfrac{\alpha z_0Dp(z_0)'}{1-\alpha+\alpha p(z_0)}\right] \leq 0$ which contradicts (7). We get $Re\dfrac{zDf(z)'}{f(z)} > 0$ for all $z \in P$ and then $f \in S^*(P)$.

The notion of α-starlikeness was introduced with the help of the generalized arithmetical mean of the functions $f(z)$ and $zDf(z)'$. We now consider a new class of functions using the generalized harmonic mean of the functions $f(z)$ and $zDf(z)'$.

Let α be a complex number. The function $f \in A$, $f(z) \neq 0$, $zDf(z)' \neq 0$ for $z \in P \setminus \{0\}$ is called α-harmonic starlike if the function $F : P \to \mathbb{C}$ defined by

$$\frac{1}{F(z)} = \frac{1-\alpha}{f(z)} + \frac{\alpha}{zDf(z)'}, \quad \text{for} \quad z \in P \tag{8}$$

is a starlike function on P.

We denote by $SH^*_\alpha(P)$ the class of α-harmonic starlike functions on P. We have $SH^*_0(P) = S^*(P)$ and $SH^*_1(P) = S^c(P)$. Using (1) and (8) it follows that a function f belongs to the class $SH^*_\alpha(P)$ if

$$Re\left[p(z) + \frac{zDp(z)'}{p(z)} - (1-\alpha)\frac{zDp(z)'}{\alpha + (1-\alpha)p(z)}\right] > 0, \quad \text{for all} \quad z \in P, \tag{9}$$

where $p(z) = \dfrac{zDf(z)'}{f(z)}$.

Theorem 3. *If $f \in SH^*_\alpha(P)$ and $\alpha \in \mathbb{C}$ with $\left|\alpha - \dfrac{1}{2}\right| \geq \dfrac{1}{2}$ then $f \in S^*(P)$.*

The proof is similar with the proof of Theorem 2.

Remark. The classes $S^*_\alpha(P)$ and $SH^*_\alpha(P)$ are the extensions of the α-starlike and α-harmonic starlike functions in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$ which were obtained by N.N. Pascu [5] and N.N. Pascu, D. Răducanu [6].

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Department of Mathematics, Faculty of Sciences, "Transilvania"
University of Braşov, Iuliu Maniu, 50, 2200 Braşov, Romania