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DATA DEPENDENCE OF THE FIXED POINTS SET OF MULTIVALUED WEAKLY PICARD OPERATORS

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Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. The purpose of this paper is to present data dependence results for some multivalued weakly Picard operatorors such as: Reich-type operators, graphic-contractions.

1. **Introduction**

The purpose of this paper is to study the following problem (see Lim [9], Rus [21], Rus-Mureşan [23], etc).

Problem. Let (X, d) be a metric space and $T_1, T_2 : X \to P(X)$ two multivalued operators. If the fixed points sets F_{T_1} and F_{T_2} are nonempty and there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$, estimate $H(F_{T_1}, F_{T_2})$, where H is the Hausdorff-Pompeiu generalized functional on $P(X)$.

Throughout the paper we follow the terminologies and the notations from Rus [20]. For the convenience of the reader, we recall some of them.

Let (X, d) be a metric space. We denote:

$$P(X) := \{A | A \text{ is a nonempty subset of } X\}, \quad P_{cl}(X) := \{A \in P(X) | A \text{ - closed}\},$$

$$P_b(X) := \{A \in P(X) |\ A - \text{bounded}\}, \quad P_{cp}(X) := \{A \in P(X) | A \text{ - compact}\},$$

$$P_{b,cl}(X) := P_b(X) \cap P_{cl}(X).$$

If $A, B \in P(X)$, then we define the functional:

$$D(A, B) := \inf\{d(a, b) | a \in A,\ b \in B\},$$

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and the following generalized functionals:

$$\rho(A,B) := \sup\{D(a,B) | a \in A\}, \quad H(A,B) := \max\{\rho(A,B), \rho(B,A)\}.$$

In this note we need the following well known properties of the functionals D and H (see Nadler [13], Reich [15], Rus [19], [20],...).

Lemma 1.1 *Let $A, B \in P(X)$ and $q \in \mathbb{R}$, $q > 1$, be given.*

Then for every $a \in A$, there exists $b \in B$ such that $d(a,b) \leq qH(A,B)$.

Lemma 1.2. *Let $A, B \in P(X)$. We suppose that there exists $\eta \in \mathbb{R}$, $\eta > 0$, such that*

(i) for each $a \in A$ there is $b \in B$ such that $d(a,b) \leq \eta$;

(ii) for each $b \in B$ there is $a \in A$ such that $d(a,b) \leq \eta$.

Then $H(A,B) \leq \eta$.

Lemma 1.3. *Let $A \in P(X)$ and $x \in X$. Then $D(x,A) = 0$ iff $x \in \overline{A}$.*

If $T : X \to P(X)$ is a multivalued operator, then we denote by F_T the fixed points set of T, i. e.

$$F_T := \{x \in X | x \in T(x)\}.$$

2. **Multivalued weakly Picard operators**

Let us start the section by recalling an important notion.

Definition 2.1. Let (X,d) be a metric space and $T : X \to P_{cl}(X)$ a multivalued operator. By definition, T is a *weakly Picard operator (*briefly *w.P.o.)* iff for all $x \in X$ and all $y \in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that*:*

*(*i*)* $x_0 = x$, $x_1 = y$,

*(*ii*)* $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$,

*(*iii*)* the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

Remark 2.2. A sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the condition *(*ii*)* and *(*iii*)*, in the Definition 2.1 is, by definition, a sequence of successive approximations of T starting from x_0.

Example 2.3. [see Rus [22]] If $t : X \to X$ is a singlevalued w.P.o., then the multivalued operator $T : X \to P_{cl}(X)$, $T(x) := \{t(x)\}$, for each $x \in X$, is a multivalued w.P.o.

Example 2.4. Let $t_i : X \to X$, $i \in \{1, 2, \ldots, n\}$, be singlevalued contractions. Then the multivalued operator $T : X \to P_{cl}(X)$, $T(x) = \{t_1(x), \ldots, t_n(x)\}$, for each $x \in X$, is a multivalued w.P.o.

Example 2.5. [see Covitz-Nadler [4] and Reich [15]] Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued contraction. Then T is a multivalued w.P.o.

Other examples will be given in the next paragraphs.

3. Data dependence of the fixed points set of Reich-type operators

The first main result of this paper is the following:

Theorem 3.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$, be two multivalued operators. We suppose that:*

(i) *there exist $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}_+$, $\alpha_i + \beta_i + \gamma_i < 1$, such that*

$$H(T_i(x), T_i(y)) \leq \alpha_i d(x, y) + \beta_i D(x, T_i(x)) + \gamma_i D(y, T_i(y)),$$

for all $x, y \in X$ and $i \in \{1, 2\}$;

(ii) *there exists $\eta > 0$ such that*

$$H(T_1(x), T_2(x)) \leq \eta, \text{ for all } x \in X.$$

Then

(a) *$F_{T_i} \in P_{cl}(X)$, $i \in \{1, 2\}$,*

(b) *the operators T_1, T_2 are w.P.o. and*

$$H(F_{T_1}, F_{T_2}) \leq \eta(1 - \min\{\gamma_1, \gamma_2\})(1 - \max\{\alpha_1 + \beta_1 + \gamma_1, \alpha_2 + \beta_2 + \gamma_2\})^{-1}.$$

Proof. (a) From a theorem of Reich (Theorem 5 in [15]), we have that $F_{T_i} \in P(X)$, $i \in \{1, 2\}$. Let us prove that the fixed points set of a multivalued operator T, satisfying a condition of type (i) (with $\alpha, \beta, \gamma \in \mathbb{R}_+$, $\alpha + \beta + \gamma < 1$) is closed. For this purpose let $x_n \in F_T$, $n \in \mathbb{N}$, such that $x_n \to x^*$, as $n \to +\infty$. We have:

$$D(x^*, T(x^*)) \leq d(x^*, x_n) + D(x_n, T(x^*)) \leq d(x^*, x_n) + H(T(x_n), T(x^*)) \leq$$

$$\leq d(x^*, x_n) + \alpha d(x_n, x^*) + \beta D(x_n, T(x_n)) + \gamma D(x^*, T(x^*)).$$

From this relation we have that

$$D(x^*, T(x^*)) \leq (1+\alpha)(1-\gamma)^{-1} d(x^*, x_n) \to 0, \text{ as } n \to \infty.$$

Hence, by Lemma 1.3, $x^* \in T(x^*)$.

(*b*) Let $q \in]1, \min\{(\alpha_1+\beta_1+\gamma_1)^{-1}, (\alpha_2+\beta_2+\gamma_2)^{-1}\}[$. Let $x_0 \in F_{T_1}$ and $x_1 \in T_2(x_0)$ such that

$$d(x_0, x_1) \leq qH(T_1(x_0), T_2(x_0)) \leq q\eta.$$

Using again Lemma 1.1, there exists $x_2 \in T_2(x_1)$ such that

$$d(x_1, x_2) \leq q(\alpha_2+\beta_2)(1-q\gamma_2)^{-1} d(x_0, x_1).$$

By induction, we prove that there exists a sequence of successive approximations of T_2, starting from $x_0 \in F_{T_1}$, such that

$$d(x_n, x_{n+1}) \leq L_2(q) d(x_{n-1}, x_n), \; n \in \mathbb{N}^*,$$

where $L_2(q) = q(\alpha_2+\beta_2)(1-q\gamma_2)^{-1} < 1$.

This relation implies that $x_n \to x^*$, as $n \to \infty$. By standard argument we prove that $x^* \in F_{T_2}$ and

$$d(x_n, x^*) \leq [1 - L_2(q)]^{-1} [L_2(q)]^n q\eta, \; n \in \mathbb{N}.$$

For $n = 0$, we obtain

$$d(x_0, x^*) \leq [1 - L_2(q)]^{-1} q\eta. \tag{1}$$

By a similar way, we have that for all $y_0 \in F_{T_2}$ and $y_1 \in T_1(y_0)$, there exists a sequence of successive approximations of T_1 such that

$$y_n \to y^* \in F_{T_1}, \text{ as } n \to \infty$$

and

$$d(y_n, y^*) \leq [1 - L_1(q)]^{-1}] [L_1(q)]^n q\eta, \; n \in \mathbb{N},$$

where $L_1(q) := q(\alpha_1+\beta_1)(1-q\gamma_1)^{-1} < 1$.

For $n = 0$, we have

$$d(y_0, y^*) \leq [1 - L_1(q)]^{-1} q\eta. \tag{2}$$

By Lemma 1.2, using (1) and (2) we have

$$H(F_{T_1}, F_{T_2}) \leq [1 - \max\{L_1(q), L_2(q)\}]^{-1} q\eta.$$

Letting $q \searrow 1$, we get the conclusion. □

Remark 3.2. For $\beta_i = \gamma_i = 0$ we have a result given by Lim *[9]*. See also Rus *[21]*.

4. Data dependence of the fixed points set of multivalued graphic-contraction-type operators

A multivalued graphic-contraction-type operator is a multivalued operator $T : X \to P_{cl}(X)$ satisfying a contraction-type condition for all $x \in X$ and $y \in T(x)$. We have:

Theorem 4.1. *Let (X, d) be a complete metric space and $T_1, T_2 : X \to P_{cl}(X)$ such that:*

(i) *there exist $\alpha_i, \beta_i \in \mathbb{R}_+$, $\alpha_i + \beta_i < 1$ such that*

$$H(T_i(x), T_i(y)) \leq \alpha_i d(x, y) + \beta_i D(y, T_i(y)),$$

for every $x \in X$, every $y \in T_i(x)$ and for $i \in \{1, 2\}$;

(ii) *there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.*

If:

(iii) *T_1, T_2 are closed multivalued operators*

or

(iv) *there exist two continuous functions $\psi_1, \psi_2 : \mathbb{R}^5_+ \to \mathbb{R}_+$ such that:*

(iv_1) $H(T_i(x), T_i(y)) \leq \psi_i(d(x, y), D(x, T_i(x)), D(y, T_i(y)), D(x, T_i(y)), D(y, T_i(x)))$, *for all $x, y \in X$ and for $i \in \{1, 2\}$;*

(iv_2) $\psi_i(0, 0, s, s, 0) < s$, *if $s > 0$, $i \in \{1, 2\}$;*

(iv_3) *If $u_1 \leq u_2$ and $v_1 \leq v_2$ then $\psi_i(u, u_1, v, w, v_1) \leq \psi_i(u, u_2, v, w, v_2)$, for all $u_i, v_i, u, v, w \in \mathbb{R}_+$ and $i \in \{1, 2\}$,*

then

(a) $F_{T_i} \in P_{cl}(X)$, *for $i \in \{1, 2\}$;*

(b) *T_i are w.P.o., for $i \in \{1, 2\}$;*

(c) $H(F_{T_1}, F_{T_2}) \leq \eta(1 - \min\{\beta_1, \beta_2\})(1 - \max\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\})^{-1}$.

Proof. Let us have (i), (ii) and (iii). From Lemma 2 in Rus [19] and (iii) we have T_i are w.P.o. and $F_{T_i} \in P(X)$, for $i \in \{1,2\}$. Let us prove that $F_{T_i} \in P_{cl}(X)$, $i \in \{1,2\}$. For this purpose, let $(x_n)_{n\in\mathbb{N}} \subset F_{T_i}$ be a convergent sequence to an element $x^* \in X$. It is sufficient to prove that $x^* \in F_{T_i}$. We have: $x_n \in T_i(x_n)$, $n \in N$. From (iii) it follows that $x^* \in T_i(x^*)$, for $i \in \{1,2\}$.

Let us have (i), (ii) and (iv). Using Theorem 1 in [19] we obtain $F_{T_i} \in P(X)$, for $i \in \{1,2\}$. Let us prove again that F_{T_i} is closed in X for each $i \in \{1,2\}$. As before, let $(x_n)_{n\in\mathbb{N}} \subset F_{T_i}$ be a convergent sequence to a point $x^* \in X$. Then:

$$D(x^*, T_i(x^*)) \le d(x^*, x_n) + D(x_n, T_i(x^*)) \le d(x^*, x_n) + H(T_i(x_n), T_i(x^*)) \le$$

$$\le d(x_n, x^*) + \psi_i(d(x_n, x^*), D(x_n, T_i(x_n)), D(x^*, T_i(x^*)), D(x_n, T_i(x^*)), D(x^*, T_i(x_n))) \le$$

$$\le d(x^*, x_n) + \psi_i(d(x_n, x^*), 0, D(x^*, T_i(x^*)), D(x_n, T_i(x^*)), d(x^*, x_n)).$$

Letting $n \to \infty$, we have:

$$D(x^*, T_i(x^*)) \le \psi_i(0, 0, D(x^*, T_i(x^*)), D(x^*, T_i(x^*)), 0).$$

From (iv_2) it follows that $D(x^*, T_i(x^*)) = 0$ and hence $x^* \in F_{T_i}$, for $i \in \{1,2\}$.

So, we get the conclusions (a) and (b). For (c) let $x_0 \in F_{T_1}$.

For every $q > 1$, there exists $x_1 \in T_2(x_0)$ such that $d(x_0, x_1) \le qH(T_1(x_0), T_2(x)) \le q\eta$. For $x_1 \in T_2(x_0)$ and $1 < q < \min\left\{\frac{1}{\alpha_1+\beta_1}, \frac{1}{\alpha_2+\beta_2}\right\}$ there is $x_2 \in T_2(x_1)$ such that $d(x_1, x_2) \le qH(T_2(x_0), T_2(x_1)) \le q[\alpha_2 d(x_0, x_1) + \beta_2 D(x_1, T_2(x_1))] \le q[\alpha_2 d(x_0, x_1) + \beta_2 d(x_1, x_2)]$ and hence

$$d(x_1, x_2) \le \frac{q\alpha_2}{1 - q\beta_2} d(x_0, x_1).$$

By induction, one prove that there exists a sequence of successive approximations for T_2, starting from $x_0 \in F_{T_1}$ such that $d(x_n, x_{n+1}) \le p_2(q) d(x_{n-1}, x_n)$, where $p_2(q) = \frac{q\alpha_2}{1 - q\beta_2} < 1$. This implies that:

1) $x_n \to x^*$, as $n \to \infty$,

2) $x^* \in F_{T_2}$,

3) $d(x_n, x^*) \le \frac{[p_2(q)]^n}{1 - p_2(q)} d(x_0, x_1) \le \frac{[p_2(q)]^n}{1 - p_2(q)} q\eta$, $n \in \mathbb{N}$.

Interchanging the roles, one can prove that for each $y_0 \in F_{T_2}$, there exists a sequence of successive approximations for T_1, starting from y_0 such that

1') $y_n \to y^*$, as $n \to \infty$,

2') $y^* \in F_{T_1}$,

3') $d(y_n, y^*) \leq \frac{[p_1(q)]^n}{1-p_1(q)} d(y_0, y_1) \leq \frac{[p_1(q)]^n}{1-p_1(q)} q\eta$, $n \in \mathbb{N}$, (where $p_1(q) = \frac{q\alpha_1}{1-q\beta_1} < 1$).

For $n = 0$ we get $d(x_0, x^*) \leq \frac{q\eta}{1-p_2(q)}$ and $d(y_0, y^*) \leq \frac{q\eta}{1-p_1(q)}$. As consequence $H(F_{T_1}, F_{T_2}) \leq q\eta[1 - \max\{p_1(q), p_2(q)\}]^{-1}$.

Letting $q \searrow 1$, the conclusion follows. □

5. **Applications**

We shall prove now a data dependence result for the following equation:

$$\phi(u) + \psi(u) = v, \quad u \in U. \tag{3}$$

Let us denote by $S_{\psi,v}$ the solutions set for (3). We have:

Theorem 5.1. *Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be real Banach spaces and let $\phi : U \to V$ be a continuous linear operator from U onto V. Put $\alpha = \sup\{\inf\{\|u\|_U | u \in \phi^{-1}(v)\}, v \in V, \|v\|_V \leq 1\}$.*

Then, for every $v_1, v_2 \in V$ and every lipschitzian operators $\psi_1, \psi_2 : U \to V$ (with the same Lipschitz constant $L > 0$) satisfying the following assertions:

i) *there is $\eta_1 > 0$ such that $\|v_1 - v_2\|_V \leq \eta_1$;*

ii) *there exists $\eta_2 > 0$ such that $\|\psi_1(u) - \psi_2(u)\|_V \leq \eta_2$, for each $u \in U$;*

iii) *$\alpha L < 1$*

are true the conclusions:

a) *$S_{\psi_i,v_i} \in P_{cl}(U)$, for $i \in \{1, 2\}$;*

b) *$H(S_{\psi_1,v_1}, S_{\psi_2,v_2}) \leq \frac{\alpha(\eta_1+\eta_2)}{1-\alpha L}$.*

Proof. From a result given by B. Ricceri (see [17], Theorem 4) it follows that $S_{\psi_i,v_i} \neq \emptyset$ and $S_{\psi_i,v_i} = Fix F_i$, where $F_i : U \to P_{cl}(U)$ is a multivalued αL-contraction, given by the formula $F_i(u) = \phi^{-1}(v_i - \psi_i(u))$, for $i \in \{1, 2\}$ (see also [18]). From Theorem 3.1 one have:

$$H(S_{\psi_1,v_1}, S_{\psi_2,v_2}) \leq \frac{1}{1-\alpha L} \sup_{u \in U} H(F_1(u), F_2(u)).$$

But $H(F_1(u), F_2(u)) = H(\phi^{-1}(v_1 - \psi_1(u)), \phi^{-1}(v_2 - \psi_2(u))) \leq \alpha\|v_1 - \psi_1(u) - v_2 + \psi_2(u)\| \leq \alpha(\eta_1 + \eta_2)$, for each $u \in U$ and hence the conclusion follows. □

Let us consider now the following functional equations of n-th order:

$$\varphi(x) \in G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))), \quad x \in X, \tag{4}$$

$$\varphi(x) \in G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))), \quad x \in X, \tag{5}$$

where φ is an unknown function and the multivalued operators G_1, G_2 and the singlevalued functions f_k, g_k $(k \in \{1, 2, \ldots, n\})$ are given. Let us denote by S_i $(i \in \{1, 2\})$ the space of continuous solutions for problems (4) and (5) respectively.

Theorem 5.2. *Let X be a compact metric space and Y be a nonempty, closed, convex subset of a Banach space. Let $G_1, G_2 : X \times Y^n \to P_{cl,cv}(Y)$ be multivalued operators and $f_k, g_k : X \to X$, $k \in \{1, 2, \ldots, n\}$ functions. We assume the following conditions on the given operators:*

i) *there exist two functions $\beta_i : \mathbb{R}_+^n \to \mathbb{R}_+$ non-decreasing with respect to each variable with the property $\beta_i(t, t, \ldots, t) \leq a_i t$, for each $t > 0$, with $0 \leq a_i < 1$ such that one have:*

$$H(G_i(x, y_1, \ldots, y_n), G_i(x, z_1, \ldots, z_n)) \leq \beta_i(\|y_1 - z_1\|, \ldots, \|y_n - z_n\|),$$

for $x \in X$, $y_k, z_k \in Y$ $(k \in \{1, 2, \ldots, n\})$ and for $i \in \{1, 2\}$;

ii) *$f_k, g_k : X \to X$ are continuous, $k \in \{1, 2, \ldots, n\}$;*

iii) *G_1, G_2 are lower semicontinuous (l.s.c.);*

iv) *there exist η_k, $\tilde{\eta} > 0$ such that $\|f_k(x) - g_k(x)\| \leq \eta_k$ for $k \in \{1, 2, \ldots, n\}$ and $H(G_1(x, y_1, \ldots, y_n), G_2(x, y_1, \ldots, y_n)) \leq \tilde{\eta}$, for $x \in X$ and $y_1, \ldots, y_n \in Y$.*

Then:

a) *$S_i \in P_{cl}(\mathcal{C})$, for $i \in \{1, 2\}$ (where $\mathcal{C} = C(X, Y)$ is the space of continuous functions from X to Y);*

b) *$H(S_1, S_2) \leq (1 - \max\{a_1, a_2\}) [\beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}]$.*

Proof. From Theorem 4.1 in Węgrzyk [26] we get that $S_i = F_{T_i}$, where $T_i : \mathcal{C} \to P_{cl,cv}(\mathcal{C})$, $i \in \{1, 2\}$ are multivalued operators given by the formulae:

$$T_1(\varphi) = \{\psi \in \mathcal{C} | \psi(x) \in G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))),\ x \in X\}$$

and

$$T_2(\varphi) = \{\psi \in \mathcal{C} |\ \psi(x) \in G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))),\ x \in X\}.$$

From Lemma 4.1 in the same paper [26], we have that $H(T_i(\varphi_1), T_i(\varphi_2)) \leq \gamma_i(\overline{d}(\varphi_1, \varphi_2))$, for $\varphi_1, \varphi_2 \in \mathcal{C}$, where $\gamma_i(t) = \beta_i(t, \ldots, t)$, for $t \in \mathbb{R}_+$ and $\overline{d}(\varphi_1, \varphi_2) = \sup\{\|\varphi_1(x) - \varphi_2(x)\| \mid x \in X\}$.

By $i)$ it follows that T_i are multivalued a_i-contractions, for $i \in \{1, 2\}$. Then, we obtain:

$$S_i \in P_{cl}(\mathcal{C}), \text{ for } i \in \{1, 2\}$$

and

$$H(S_1, S_2) = H(F_{T_1}, F_{T_2}) \leq [1 - \max\{a_1, a_2\}] \sup_{\varphi \in \mathcal{C}} H(T_1(\varphi), T_2(\varphi)). \tag{6}$$

On the other side, let us estimate $H(T_1(\varphi), T_2(\varphi))$.

For this purpose, let $\varphi_1 \in T_1(\varphi)$. Then $\varphi_1(x) \in G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x)))$, $x \in X$. We have

$$D(\varphi_1(x), G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) \leq H(G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))),$$

$$G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) \leq H(G_1(x, \varphi(f_1(x)), \ldots, \varphi(f_n(x))),$$

$$G_1(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) + H(G_1(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))),$$

$$G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x)))) \leq \beta(\|\varphi(f_1(x)) - \varphi(g_1(x))\|, \ldots, \|\varphi(f_n(x)) - \varphi(g_n(x))\|) + \tilde{\eta}.$$

From the uniform continuity of φ on the compact space X and from iv) we get that

$$\|\varphi(f_k(x)) - \varphi(g_k(x))\| \leq \eta_k, \text{ for each } x \in X.$$

Hence we conclude that

$$D(\varphi_1(x), G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x))) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta},$$

for each $x \in X$.

Then, for a fixed $\varepsilon > 0$ and for every $x \in X$ there exists $z_x \in G_2(x, \varphi(g_1(x)), \ldots, \varphi(g_n(x)))$ such that

$$\|\varphi_1(x) - z_x\| \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta} + \varepsilon.$$

Using the same argument like in the proof of Lemma 4.1 from [26] we infer that for every $\varepsilon > 0$ there exists a continuous function $\varphi_2 \in T_2(\varphi)$ such that

$$\overline{d}(\varphi_1, \varphi_2) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta} + \varepsilon.$$

It follows $D(\varphi_1, T_2(\varphi)) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}$. From the analogous inequality: $D(\varphi_2, T_1(\varphi)) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}$, for every $\varphi_2 \in T_2(\varphi)$ we get that

$$H(T_1(\varphi), T_2(\varphi)) \leq \beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}.$$

Making use of the estimate (6), we obtain

$$H(S_1, S_2) \leq (1 - \max\{a_1, a_2\}) [\beta(\eta_1, \ldots, \eta_n) + \tilde{\eta}]. \quad \square$$

Remark 5.3. For other applications see *[2], [3], [7], [8], [11], [24]*.

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