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NEW UNIVALENCE CRITERION FOR CERTAIN INTEGRAL OPERATOR

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Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. In this work we prove a new univalence criterion for the analicity and univalence in the unit disc $U = \{z \in C : |z| < 1\}$ of an integral operator.

1. INTRODUCTION

Let A be the class of the functions f which are analytic in the unit disc and $f(0) = f'(0) - 1 = 0$. We denote by S the class of the functions $f \in A$ which are univalent in U.

In the theory of univalent functions an interesting problem is to find those integral operators which preserve the univalence of the class S.

Many authors studied the problem of integral operators which preserve the class S. In this sense, important results are due to Y. J. Kim, E.P. Merkes [1], M. Nunokawa [3] and J. Pfaltzgraff [5].

2. PRELIMINARIES

We will need the following theorem in this paper.

THEOREM A[4]. Let α be a complex number, $Re\alpha > 0$ and $f \in A$.

If

$$\left(1 - |z|^{2Re\alpha}\right) \left| \frac{zf''(z)}{f'(z)} \right| \leq Re\alpha \tag{1}$$

for all $z \in U$, then the function

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$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u)du\right]^{\frac{1}{\alpha}} \tag{2}$$

is in the class S.

3. MAIN RESULT

THEOREM. Let $g \in S$ and $\alpha = a + bi$ be a complex number and $a \in (0, 4]$. If

$$a^4 + a^2b^2 - 4 \geq 0, a \in \left(0, \frac{1}{2}\right) \, and \quad a^2 + b^2 - 16 \geq 0, \quad a \in \left[\frac{1}{2}, 4\right] \tag{3}$$

then the function

$$H_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\alpha}} \tag{4}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\alpha}} du. \tag{5}$$

The function f is regular in U.From (5) we have
$f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}, f''(z) = (\frac{1}{\alpha} \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}-1} \frac{zg'(z)-g(z)}{z^2}$
and

$$\frac{1-|z|^{2a}}{a} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}} \left(\frac{z(g'(z))}{g(z)} + 1\right). \tag{6}$$

for all $z \in U$.

From (6) we obtain

$$\frac{1-|z|^{2a}}{a} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2a}}{a\sqrt{a^2+b^2}} \left(\frac{1+|z|}{1-|z|} + 1\right). \tag{7}$$

and hence we get

$$\frac{1-|z|^{2a}}{a} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{2}{a\sqrt{a^2+b^2}} \frac{1-|z|^{2a}}{1-|z|} \tag{8}$$

for all $z \in U$.

Let us note $|z| = x$, $x \in (0,1)$ and $\phi(x) = \frac{1-x^{2a}}{1-x}, a > 0$. It easy to prove that

$$\phi(x) \leq \begin{cases} 1 & \text{if} \quad a \in \left(0, \frac{1}{2}\right) \\ 2a & \text{if} \quad a \in [\frac{1}{2}, \infty) \end{cases} \tag{9}$$

Using $a \in (0,4]$ and the relations (8),(9),(3) we obtain

$$\left(\frac{1-|z|^{2a}}{a}\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{10}$$

for all $z \in U$.

From (5) we have $f'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\alpha}}$ and using (10) by Theorem A it results that the function H_α is in the class S.

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