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NON-ANALYTIC FUNCTIONS IN AN ELLIPSE

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Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

The purpose of this paper is to generalize some results about functions of class C^1 on the unit disc obtained by P.T.Mocanu in [1], considering functions of class C^1 on an elliptic domain. We also obtained a sufficient condition for univalency, by introducing the notion of starlikeness with respect to the origin for functions of class C^1 on the elliptic domain.

Let E denote the elliptic domain
$E := \left\{ z = x + iy \in C : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0 \right\}$.

Consider a complex function defined on E of the form $f(z) = u(x, y)+iv(x, y)$. For $r \in (0, 1)$ and $\theta \in [0, 2\pi]$, the elliptic coordinates of a point $z = x + iy$ from E are
$\begin{cases} x = ar\cos\theta \\ y = br\sin\theta. \end{cases}$

Definition 1. The function $f : E \to C$ is said to be of class $C^1(E)$ if the real functions $u = Ref$ and $v = Imf$, of the real variables $x = Rez$, $y = Imz$, are continuous and have continuous first order partial derivatives in E.

For $f \in C^1(E)$, we denote

$$Df(z) = z\frac{\partial f}{\partial z} - \overline{z}\frac{\partial f}{\partial \overline{z}} \tag{1}$$

$$\mathcal{D}f(z) = \frac{z(a^2+b^2) - \overline{z}(a^2-b^2)}{2ab}\frac{\partial f}{\partial z} + \frac{\overline{z}(a^2+b^2) - z(a^2-b^2)}{2ab}\frac{\partial f}{\partial \overline{z}} \tag{2}$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2ab}\left(a^2\frac{\partial}{\partial x} - ib^2\frac{\partial}{\partial y}\right)$$

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and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2ab}\left(a^2\frac{\partial}{\partial x} + ib^2\frac{\partial}{\partial y}\right).$$

The linear differential operators defined by (1) and (2) verify the rules of the differential calculus, for example:

$D(f+g) = Df + Dg,$

$D(fg) = fDg + gDf,$

$D(\frac{f}{g}) = \frac{gDf - fDg}{g^2},$

$D(f \circ g) = \frac{\partial f}{\partial g}Dg + \frac{\partial f}{\partial \overline{g}}D\overline{g};$

For $a = 1$ and $b = 1$, from (1) and (2), we obtain the differential operators defined in [Mo].

The two operators have the following properties:

$$\begin{array}{ll}
D\overline{f} = -\overline{Df} & \mathcal{D}\overline{f} = \overline{\mathcal{D}f} \\
D\operatorname{Re} f = i\operatorname{Im} Df & \mathcal{D}\operatorname{Re} f = \operatorname{Re}\mathcal{D}f \\
D\operatorname{Im} f = -i\operatorname{Re} Df & \mathcal{D}\operatorname{Im} f = \operatorname{Im}\mathcal{D}f \\
D\,|f| = i\,|f|\operatorname{Im}\frac{Df}{f} & \mathcal{D}\,|f| = |f|\operatorname{Re}\frac{\mathcal{D}f}{f} \\
D\arg f = -i\operatorname{Re}\frac{Df}{f} & \mathcal{D}\arg f = \operatorname{Im}\frac{\mathcal{D}f}{f}
\end{array}$$

We also have:

$$\frac{\partial f}{\partial \theta} = iDf \qquad \text{and} \qquad \frac{\partial f}{\partial r} = \frac{1}{r}Df$$

where $z = r(a\cos\theta + ib\sin\theta)$.

From here we deduce that

$$\frac{\partial\,|f|}{\partial \theta} = -\,|f|\operatorname{Im}\frac{Df}{f} \qquad \text{and} \qquad \frac{\partial\,|f|}{\partial r} = \frac{|f|}{r}\operatorname{Re}\frac{\mathcal{D}f}{f} \tag{3}$$

$$\frac{\partial}{\partial \theta}\arg f = \operatorname{Re}\frac{Df}{f} \qquad \text{and} \qquad \frac{\partial}{\partial r}\arg f = \frac{1}{r}\operatorname{Re}\frac{\mathcal{D}f}{f} \tag{4}$$

The Jacobian of the function $f \in C^1(E)$ is given by

$$Jf = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \overline{z}}\right|^2 .$$

It is known that a function verifying $Jf > 0$, $z \in E$, is locally univalent and preserves the orientation.

Definition 2. The continuous function $f : E \to C$, $f(0) = 0$, is called starlike in E with respect to the origin if it is univalent in E and $f(E)$ is a starlike set.

Theorem 3. *A functionf $\in C^1(E)$ that satisfies the conditions:*

(i) $f(0) = 0$ and $f(z) \neq 0$ for all $z \in E\backslash\{0\}$,

(ii)$Jf(z) > 0$ for all $z \in E$,

(iii)$\mathrm{Re}\,\frac{Df(z)}{f(z)} > 0$ *for all $z \in E\backslash\{0\}$,*

is starlike in E.

Proof. We denote $E_r := \left\{z = x + iy \in C : \frac{x^2}{(ar)^2} + \frac{y^2}{(br)^2} - 1 < 0\right\}$ and $C_{\mathrm{r}} = f(\partial E_{\mathrm{r}})$ for $r \in (0,1)$. From (4) and (iii) we deduce that

$$\frac{\partial}{\partial\theta} \arg f(r(a\cos\theta + ib\sin\theta)) > 0, \text{ for all } \theta \in [0, 2\pi] \text{ and all } r \in (0,1).$$

Therefore C_r is a starlike curve (not necesarry simple) with respect to the origin, for all $r \in (0,1)$.

In order to prove the univalency of f it is enough to show that C_r are Jordan curves and they are each two disjoint. From the condition (i) follows that the curves $C_r, r \in (0,1)$, are homotopic in $C\backslash\{0\}$, therefore the index of C_r with respect to the origin is the same, for each $r \in (0,1)$, i.e. $n(C_r, 0) = const$. Because of the condition (ii) there exists a neighbourhood of the origin such that f is univalent and preserves orientation in this neighbourhood. Thus we have an $r_0 \in (0,1)$ such that for every $r < r_0$, $n(C_r, 0) = 1$, meaning that the variation of the argument along C_r is 2π. We conclude that C_r is a Jordan curve, for each $r \in (0,1)$.

In order to prove that every two different curves C_r and $C_{r'}$ are disjoint, we will show that for any ray starting from the origin, the modulus of the unique point of intersection of this ray with the curve C_r is a strictly increasing function of r, as r increases in the interval $(0,1)$.

Let us fix $\varphi \in (0, 2\pi)$. The system

$$\begin{cases} \arg f(z) = \varphi \\ z = r(a\cos\theta + ib\sin\theta) \end{cases}, r \in (0,1)$$

has a unique solution $\theta = \theta(r)$, that gives us the unique point $z = r(a\cos\theta + ib\sin\theta)$. For this value of z we consider

$$R(r) = |f(z)| \tag{5}$$

We will show that $R(r)$ is strictly increasing in $(0,1)$.

From (5), by differentiating with respect to r, we get

$$\frac{dR}{dr} = R\left(\frac{1}{r}\operatorname{Re}\frac{\mathcal{D}f}{f} - \frac{d\theta}{dr}\operatorname{Im}\frac{Df}{f}\right). \tag{6}$$

From the relation $\arg f(z) = \varphi$, we obtain

$$\frac{1}{r}\operatorname{Im}\frac{\mathcal{D}f}{f} + \frac{d\theta}{dr}\operatorname{Re}\frac{Df}{f} = 0. \tag{7}$$

By eliminating $\frac{d\theta}{dr}$ between the equations (6) and (7) we get

$$\frac{dR}{dr}\operatorname{Re}\frac{Df}{f} = \frac{R}{r}\left(\operatorname{Re}\frac{Df}{f}\operatorname{Re}\frac{\mathcal{D}f}{f} + \operatorname{Im}\frac{Df}{f}\operatorname{Im}\frac{\mathcal{D}f}{f}\right)$$

or

$$\frac{dR}{dr}\operatorname{Re}\frac{Df}{f} = \frac{1}{r}\operatorname{Re}\left(Df\overline{\mathcal{D}f}\right)$$

A simple calculus shows that $\operatorname{Re}\left(Df\overline{\mathcal{D}f}\right) = abr^2Jf$, therefore

$$\frac{dR}{dr}\operatorname{Re}\frac{Df}{f} == abrJf,$$

Because $\frac{dR}{dr} > 0$, R is a strictly increasing function in $(0,1)$. We proved the univalency of f.

We have that the domain $f(U_r)$ is starlike for each $r \in (0,1)$ and $f(U_r) \subset f(U_{r\prime})$ for $0 < r < r\prime < 1$. It follows that $f(U)$ is also a starlike domain. Our theorem is proved.

References

[1] **Petru T. Mocanu**, *Starlikeness and convexity for non-analytic functions in the unit disc*, Mathematica 22(45), 1, 1980, 77-83.

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