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ON CONVEX FUNCTIONS IN AN ELLIPTICAL DOMAIN

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Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. In this note we define the notions of convexity for analytic functions in the ellipse $E = \left\{ z = x + iy \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 < 0 \right\}$, $a > b > 0$. We obtain sufficient conditions for an analytic function to be a convex function in the ellipse E.

1. Introduction and preliminaries

Let g be a complex function defined in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For $z = x + iy \in U$ we consider $u(x, y) = \mathrm{Re}g(z)$ and $v(x, y) = \mathrm{Im}g(z)$. The function g belongs to the class $C^1(U)$, respectively $C^2(U)$ if the functions u and v of the real variables x and y have continuous first order, respectively second order, partial derivatives in U [1].

For $g \in C^1(U)$ the following operators are defined

$$Dg(z) = z\frac{\partial g}{\partial z} - \overline{z}\frac{\partial g}{\partial \overline{z}} \quad \text{and} \quad Jg = \left|\frac{\partial g}{\partial z}\right|^2 - \left|\frac{\partial g}{\partial \overline{z}}\right|^2$$

where

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) \quad \text{and} \quad \frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right).$$

P.T. Mocanu [1] obtained sufficient conditions for a non-analytic function in the unit disc, to be univalent and convex.

Definition 1. [1] A function g of the class $C^1(U)$ is a convex function in U if it is univalent and $g(U)$ is a convex domain.

A sufficient condition for convexity is given in the following theorem.

Theorem 1. [1] *If the function $g \in C^1(U)$ satisfies the conditions*

(i) $g(0) = 0$, $Dg \in C^1(U)$ and $g(z)Dg(z) \neq 0$, for all $z \in U \setminus \{0\}$,

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(ii) $Jg(z) > 0$, *for all* $z \in U$

(iii) $\mathrm{Re}\dfrac{D^2g(z)}{Dg(z)} > 0$, *for all* $z \in U \setminus \{0\}$

then g *is a convex function in* U.

2. **Main results**

Let f be an analytic function in the ellipse E.

Definition 2. The function f is a convex function in E if it is an univalent function in E and $f(E)$ is a convex domain.

In the next two theorems, sufficient conditions for an analytic function in E to be convex in E, are given.

Theorem 2. *If the analytic function* $f : E \to \mathbb{C}$ *satisfies the conditions*

(i) $f(0) = 0$ *and* $f'(z) \neq 0$, *for all* $z \in E$,

(ii) the inequality

$$(a^2 + b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] - (a^2 - b^2)\mathrm{Re}\left[\frac{\overline{z}f''(z)}{f'(z)} + 1\right] > 0 \tag{1}$$

holds for all $z \in E$, *then* f *is a convex function in* E.

Proof. Let $h : U \to E$ be the function defined by

$$h(z) = \frac{a+b}{2}z + \frac{a-b}{2}\overline{z}. \tag{2}$$

Then h belongs to the class $C^1(U)$, is an univalent function in U and $h(U) = E$.

We consider the functions $g : U \to \mathbb{C}$, $g = f \circ h$. In order to prove that f is a convex function in E it is sufficient to show that the function g satisfies the conditions from theorem 1. We have

$$Dg(z) = f'(u)\left(\frac{a+b}{2}z - \frac{a-b}{2}\overline{z}\right) \tag{3}$$

where $u = h(z) \in E$. Since $f'(u) \neq 0$, for all $u \in E$, then $g(z)Dg(z) \neq 0$, for all $z \in U \setminus \{0\}$. The Jacobian of g is

$$Jg(z) = ab|f'(u)|^2 > 0, \quad \text{for all} \quad z \in U.$$

We also have

$$\frac{D^2g(z)}{Dg(z)} = \frac{f''(u)}{f'(u)}\left(\frac{a+b}{2}z - \frac{a-b}{2}\overline{z}\right) + \frac{(a+b)z + (a-b)\overline{z}}{(a+b)z - (a-b)\overline{z}}. \tag{4}$$

From $u = \dfrac{a+b}{2}z + \dfrac{a-b}{2}\overline{z}$ and $\overline{u} = \dfrac{a-b}{2}z + \dfrac{a+b}{2}\overline{z}$ we obtain

$$z = \frac{1}{2ab}[(a+b)u - (a-b)\overline{u}] \tag{5}$$

and hence $\mathrm{Re}\dfrac{D^2g(z)}{Dg(z)} > 0$, for all $z \in U$, holds only if

$$(a^2+b^2)\mathrm{Re}\left[\frac{uf''(u)}{f'(u)}+1\right] - (a^2-b^2)\mathrm{Re}\left[\frac{\overline{u}f''(u)}{f'(u)}+1\right] > 0, \quad \text{for all} \quad u \in E.$$

Remark. For $a = b$ ($E = U$), the conditions from above are the same with the well-known conditions for convexity for analytic functions in the unit disc.

Theorem 3. *If the analytic function $f : E \to \mathbb{C}$ satisfies the conditions*

(i) $f(0) = 0$ and $f'(z) \neq 0$, for all $z \in E$,

(ii) the inequalities

$$\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right] > \frac{1}{2} \tag{6}$$

and

$$\left|\arg\left[\frac{zf''(z)}{f'(z)}+1\right]\right| \leq \arccos\frac{3(a^2-b^2)}{a^2+b^2} \tag{7}$$

are true, for all $z \in E$, then f is a convex function in E.

Proof. In order to prove that the function f is convex in E it is sufficient to show that the inequality (1) is true. From (6) we have

$$\left|\frac{zf''(z)}{f'(z)}+1\right| \geq \left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{\overline{z}f''(z)}{f'(z)}\right| \geq \mathrm{Re}\frac{\overline{z}f''(z)}{f'(z)} \tag{8}$$

and

$$\left|\frac{zf''(z)}{f'(z)}+1\right| > \frac{1}{2}, \tag{9}$$

for all $z \in E$.

From (17) we also have

$$\frac{\mathrm{Re}\left[\dfrac{zf''(z)}{f'(z)}+1\right]}{\left|\dfrac{zf''(z)}{f'(z)}+1\right|} > \frac{3(a^2-b^2)}{a^2+b^2}, \tag{10}$$

for all $z \in E$.

Using the inequalities (8), (9) and (10) we obtain

$$(a^2+b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right] - (a^2-b^2\mathrm{Re}\left[\frac{\overline{z}f''(z)}{f'(z)}+1\right] >$$

$$> (a^2+b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]-(a^2-b^2)\left[\left|\frac{zf''(z)}{f'(z)}\right|+1\right]>$$

$$> (a^2+b^2)\mathrm{Re}\left[\frac{zf''(z)}{f'(z)}+1\right]-(a^2-b^2)\left[\left|\frac{zf''(z)}{f'(z)}+1\right|+1\right]>$$

$$> 3(a^2-b^2)\left|\frac{zf''(z)}{f'(z)}+1\right|-(a^2-b^2)\left[\left|\frac{zf''(z)}{f'(z)}+1\right|+1\right]>0,$$

for all $z \in E$.

References

[1] P.T. Mocanu, *Starlikeness and convexity for non-analytic functions in the unit disc*, Mathematica 22(45), no.1(1980), 77-83.
[2] W.C. Royster, *Convexity and starlikeness of analytic functions*, Duke Math., 19(1952), 447-457.

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