STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

ON UNIVALENT FUNCTIONS IN A HALF-PLANE

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Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. A basic result in the theory of univalent functions is well-known inequality $\left|-2\left|z^2\right| + \left(1-|z|^2\right) zf''(z)/f'(z)\right| \leq 4|z|$ where f is an univalent function in the unit disc. In this note we obtain a similar result for univalent functions in the upper half-plane.

1. Introduction.

Let U be the unit disc $\{z : z \in C, |z| < 1\}$ and let A be the class of analytic and univalent functions in U. We denote by S the class of the functions f, $f \in A$, normalized by conditions $f(0) = f'(0) - 1 = 0$.

As a corollary of the inequality of the second coefficient, for the functions in the class S, it results the following well-known theorem:

Theorem A. If the function f belongs to the class A, then for all $z \in U$ we have

$$\left|-2|z|^2 + \left(1-|z|^2\right) zf''(z)/f'(z)\right| \leq 4|z|.$$

The Theorem A is the starting point for solving some problems (distortion theorem, rotation theorem) in the class S.

We denote by D the upper half-plane $\{z : Im(z) > 0\}$ and by S_D the class of analytic and univalent functions in the domain D. In this note we obtain a result, similar to the Theorem A, for functions in the class S_D.

Paper presented at the International Conference on Complex Analysis and The 8th Romanian-Finnish Seminar, Iaşi, 23-27 August 1999.

2. Main results.

Let $g : U \to D$ be the function defined from

$$g(z) = i\frac{1-z}{1+z} \tag{1}$$

The function g belongs to the class A and $g(U) = D$.

We denote by D_r the disc $g(U_r)$, where $r \in (0,1]$, $U_r = \{z : |z| < r\}$ and $U_1 = U$. We observe that, for all $0 < r < s \leq 1$ we have $D_r \subset D_s \subset D_1 = D$ and hence for all $\xi \in D$, there exists $r_0 \in (0,1)$ such that $\xi \in D_r$, for all $r \in (r_0, 1)$.

Let ξ_r and R_r be the numbers defined from

$$\xi_r = i\frac{1+r^2}{1-r^2}, \quad R_r = \frac{2r}{1-r^2} \tag{2}$$

For $\xi = g(z)$, where $|z| = r$, we have

$$|\xi - \xi_r|^2 = \frac{4\left|z + r^2\right|^2}{|1+z|^2\left(1-r^2\right)^2} \tag{3}$$

Because for all $z,\ |z| = r < 1$, we have

$$\left|z + r^2\right| = |r + rz| \tag{4}$$

it result that

$$|\xi - \xi_r| = R_r \tag{5}$$

and hence D_r is the disc with the center at the point ξ_r and the radius R_r.

Lemma. For all fixed point $\xi \in D$ there exists $r_0 \in (0,1)$ and $u_r \in U$ such that for all $r \in (r_0, 1)$

$$\xi = \xi_r + R_r u_r \tag{6}$$

and

$$\lim_{r\to 1} u_r = -i, \quad \lim_{r\to 1}[R_r(1 - |u_r|)] = \operatorname{Im}(\xi). \tag{7}$$

Proof. If $\xi \in D$, then $\left|g^{-1}(\xi)\right| < 1$ and hence for all r_0, $\left|g^{-1}(\xi)\right| < r_0 < 1$ we have $\xi \in D_r$, for all $r,\ r_0 < r < 1$.

For $x_r = \operatorname{Re}(u_r)$, $y_r = \operatorname{Im}(u_r)$, $X = \operatorname{Re}(\xi)$, $Y = \operatorname{Im}(\xi)$ we have

$$X = x_r\frac{2r}{1-r^2}, \quad Y = \frac{1+r^2}{1-r^2} + y_r\frac{2r}{1-r^2} \tag{8}$$

for all $r, r_0 < r < 1$ and hence

$$\lim_{r\to 1} x_r = \lim_{r\to 1}\frac{\left(1-r^2\right)X}{2r} = 0, \quad \lim_{r\to 1} y_r = \lim_{r\to 1}\frac{\left(1-r^2\right)Y - 1 - r^2}{2r} = -1 \tag{9}$$

From (8) we have

$$\left(1-|u_r|^2\right) R_r = \left[1-\frac{\left(1-r^2\right)^2 X^2+\left(\left(1-r^2\right) Y-\left(1+r^2\right)\right)^2}{4r^2}\right]\cdot\frac{2r}{1-r^2} \tag{10}$$

It result that

$$\lim_{r\to 1}\left(1-|u_r|^2\right) R_r = \lim_{r\to 1}\frac{2\left(1+r^2\right)\mathrm{Im}\,(\xi)-\left(1-r^2\right)|\xi|^2-1+r^2}{2r} = 2\mathrm{Im}\,(\xi) \tag{11}$$

and hence

$$\lim_{r\to 1}[(1-|u_r|)\,R_r] = \mathrm{Im}\,(\xi) \tag{12}$$

Theorem. If the function f is analytic and univalent in the domain D, for all $\xi \in D$ we have

$$\left|i-\mathrm{Im}\,(\xi)\,\frac{f''(\xi)}{f'(\xi)}\right| \le 2 \tag{13}$$

Proof. Let ξ be a fixed point in the domain D. From Lemma it result that there exists $r_0 \in (0,1)$ such that $\xi \in D_r$ for all $r \in (r_0, 1)$. We consider the function $g_r : U \to C$ defined from

$$g_r(u) = f(\xi_r + R_r u) \tag{14}$$

where $r \in (r_0, 1)$.

For all fixed $r, r \in (r_0, 1)$ the function g_r is analytic and univalent in U and from Theorem A it result that

$$\left|-2|u|^2+\left(1-|u|^2\right) R_r\frac{uf''(\xi_r+R_r u)}{f'(\xi_r+R_r u)}\right| \le 4|u| \tag{15}$$

From Lemma it result that for fixed point $\xi \in D$ there exists $u_r \in U$ such that $\xi = \xi_r + R_r u_r$ and hence, from (15) we obtain

$$\lim_{r\to 1}\left|-2|u_r|^2+\left(1-|u_r|^2\right) R_r\frac{u_r f''(\xi)}{f'(\xi)}\right| \le 4\lim_{r\to 1}|u_r| \tag{16}$$

Because $\lim_{r\to 1} u_r = -i$ and $\lim_{r\to 1}[(1-|u_r|)\,R_r] = \mathrm{Im}\,(\xi)$, form (16) we obtain the inequality (13).

Remark. The function f defined from

$$f(\xi) = \xi^2 \tag{17}$$

is analytic and univalent in the domain D and

$$\left|i-\mathrm{Im}\,(\xi)\,\frac{f''(\xi)}{f'(\xi)}\right| = \left|i-\mathrm{Im}\,(\xi)\,\frac{1}{\xi}\right| \tag{18}$$

If we observe that $\left| i - \mathrm{Im}\,(\xi)\,\dfrac{1}{\xi} \right| = 2$ for $\xi = i$, it result that the inequality (13) is best possible.

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