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ON THE UNIVALENCE OF CONVEX FUNCTIONS OF COMPLEX ORDER

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Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. In this note we study the univalence of the functions f who belong to the class of convex functions of complex order introduced by Nasr and Aouf [2]. The results obtained improve the results from paper [3].

1. **Introduction**

Let A be the class of functions f analytic in the unit disk $U = \{z \in C: \ |z| < 1\}$ and such that $f(0) = 0, \ f'(0) = 1$.
Let S denote the class of functions $f \in A, \ f$ univalent in U.
Nasr and Aouf defined the class of functions $f \in A, \ f'(z) \neq 0$ in U, for which $Re[1 + zf''(z)/(\alpha f'(z))] > 0$, where $\alpha \in C$. For a fixed complex number α, $\alpha \neq 0$, let us denote this class by $N(\alpha)$,

$$N(\alpha) = \left\{ f \in A: \ Re\left(1 + \frac{1}{\alpha}\frac{zf''(z)}{f'(z)}\right) > 0, \quad f'(z) \neq 0, \quad (\forall) z \in U \right\} \tag{1}$$

Theorem 1.1 ([3]). *Let α be a complex number, $\alpha \neq 0$ and let $f \in N(\alpha)$. If $\alpha \in D$, where*

$$D = D_1 \cup D_2 \cup [-1/2, \ -1/4] \cup [1/4, \ 3/2] \quad \textit{and} \tag{2}$$

$$D_1 = \{\alpha \in C: \quad |\alpha| \leq 1/4\}$$

$$D_2 = \{\alpha \in C: \quad |\alpha - 1/2| \leq 1/2 \quad \textit{and} \quad \pi/3 \leq |\arg \alpha| \leq \pi/2\},$$

then the function f is univalent in U.

2. Preliminaries

Theorem 2.1 ([4]). *Let $f \in A$. Let α, β, c be complex numbers, $Re\beta > 0$, $Re(2\alpha+\beta) > 0$, $Re\frac{\alpha}{\beta} > -1/2$, $|c(\alpha+\beta)+\alpha| + |\alpha| \le |\alpha+\beta|$. If there exists an analytic function g, $g \in A$, such that*

$$\left|(1+c)\frac{f'(z)}{g'(z)} - 1\right| < 1, \qquad (\forall) z \in U,$$

$$\left|\left[(1+c)\frac{f'(z)}{g'(z)} - 1\right] |z|^{2(\alpha+\beta)} + \frac{1-|z|^{2(\alpha+\beta)}}{\alpha+\beta}\left(\frac{zg''(z)}{g'(z)} - \alpha\right)\right| \le 1$$

for all $z \in U \setminus \{0\}$, then the function

$$F(z) = \left(\beta \int_0^z u^{\beta-1} f'(u) du\right)^{1/\beta}$$

is analytic and univalent in U.

The results obtained are proved by using Theorem 2.1 in the particular case $f \equiv g$ and $\alpha = 1-\beta$. For this choise, from Theorem 2.1 we get the following

Corollary 2.1. *Let $f \in A$ and let β, c be complex numbers. If $|\beta - 1| < 1$, $|c| < 1$, $|c+1-\beta| + |\beta-1| \le 1$ and*

$$\left|c|z|^2 + (1-|z|^2)\left(\frac{zf''(z)}{f'(z)} + \beta - 1\right)\right| \le 1, \qquad (\forall) z \in U, \tag{3}$$

then the function

$$F(z) = \left(\beta \int_0^z u^{\beta-1} f'(u) du\right)^{1/\beta} \tag{4}$$

is analytic and univalent in U.

Theorem 2.2 ([1]). *If g is a starlike function in U and $-1/2 \le \alpha \le 3/2$, then the function*

$$G(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha} du$$

is a close-to-convex function in U.

Lemma 2.1. *If g is a starlike function in U and a is a fixed point from the unit disk U, then the function*

$$h(z) = \frac{a \cdot z}{(a+z)(1+\overline{a}z)g(a)} \cdot g\left(\frac{a+z}{1+\overline{a}z}\right) \tag{5}$$

is a starlike function in U.

3. **Main results**

Theorem 3.1. *Let α, β be complex numbers, $\alpha \neq 0$, $|\beta - 1| < 1$ and let $f \in N(\alpha)$. If*

$$|\alpha| < \frac{1 - |\beta - 1|}{2}, \tag{6}$$

then it exists an univalent function F in U, such that

$$f(z) = \int_0^z \left(\frac{F(u)}{u}\right)^{\beta-1} F'(u)du \ , \quad z \in U. \tag{7}$$

Proof. Let us consider the function

$$g(z) = z \cdot (f'(z))^{1/\alpha} \ , \ \ \alpha \neq 0.$$

We have

$$\frac{zg'(z)}{g(z)} = 1 + \frac{1}{\alpha}\frac{zf''(z)}{f'(z)} \tag{8}$$

Because $f \in N(\alpha)$ it follows that $Re[zg'(z)/g(z)] > 0$ in U and hence g is a starlike function in U. Let h be the function defined by (5), $h(z) = z + a_2z^2 + \dots$. We obtain

$$a_2 = \frac{h''(0)}{2} = (1 - |a|^2)\frac{g'(a)}{g(a)} - \frac{1 + |a|^2}{a}$$

and then

$$\frac{zg'(z)}{g(z)} = \frac{1 + a_2z + |z|^2}{1 - |z|^2} \tag{9}$$

The relations (8) and (9) lead to

$$\frac{zf''(z)}{f'(z)} = \alpha\left(\frac{zg'(z)}{g(z)} - 1\right) = \alpha\frac{a_2z + 2|z|^2}{1 - |z|^2} \tag{10}$$

Taking into account (10) it results

$$c|z|^2 + (1 - |z|^2)\left(\frac{zf''(z)}{f'(z)} + \beta - 1\right) = \tag{11}$$

$$= (c + 2\alpha + 1 - \beta)|z|^2 + \alpha a_2 z + \beta - 1 \ .$$

If $c = \beta - 1 - 2\alpha$, from (6) it follows that $|c| < 1$ and also

$$|c + 1 - \beta| + |\beta - 1| = |2\alpha| + |\beta - 1| < 1 \ .$$

Since h is a starlike function, then $|a_2| \leq 2$ and in view of (6) , the relation (11) becomes

$$\left| c|z|^2 + (1 - |z|^2)\left(\frac{zf''(z)}{f'(z)} + \beta - 1\right) \right| =$$

$$= |\alpha a_2 z + \beta - 1| \leq 2|\alpha| + |\beta - 1| < 1 .$$

From Corollary 2.1 we conclude that the function

$$F(z) = \left(\beta \int_0^z u^{\beta-1} f'(u) du\right)^{1/\beta}$$

is analytic and univalent in U.

We have $F^{\beta-1}(z)F'(z) = z^{\beta-1}f'(z)$ and therefore

$$f'(z) = \left(\frac{F(z)}{z}\right)^{\beta-1} F'(z).$$

It follows that the function f is given by (7), where F is analytic and univalent in U.

If in Theorem 3.1 we take $\beta = 1$, then we have $f(z) = F(z)$ and we get the following result

Corollary 3.1. *Let α be a complex number, $\alpha \neq 0$ and let $f \in N(\alpha)$. If $|\alpha| < 1/2$, then the function f is univalent in U.*

Theorem 3.2. *Let α be a complex number , $\alpha \neq 0$ and let $f \in N(\alpha)$. If $\alpha \in D$, where*

$$D = D_1 \cup [1/2,\ 3/2] \cup \{-1/2\} , \tag{12}$$

$$D_1 = \{\alpha \in C : \ |\alpha| < 1/2\} ,$$

then the function f is univalent in U.

If α is a real number, $\alpha \notin D$, then the function

$$f(z) = \int_0^z (1-u)^{-2\alpha} du \tag{13}$$

belongs to the class $N(\alpha)$ but it is not univalent in U.

Proof. If $\alpha \in D_1$, from Corollary 3.1 it follows that f is an univalent function. Let α be a real number, $\alpha \in [-1/2,\ 3/2] \setminus \{0\}$. In the same manner as in Theorem 3.1 we consider the function $g(z) = z(f'(z))^{1/\alpha}$. The function g being a starlike function, from Theorem 2.2 it follows that the function

$$G(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\alpha} du = \int_0^z f'(u) du = f(z)$$

is a close-to-convex function. For the function f defined by (13) a short computation gives

$$1 + \frac{1}{\alpha}\frac{zf''(z)}{f'(z)} = \frac{1+z}{1-z}$$

For $z \in U$ we have $Re(1+z)/(1-z) > 0$ and hence $f \in N(\alpha)$.

For $\beta \in R,\ \beta \neq 0$, we know that the function $h(z) = (1-z)^{\beta}$ is univalent in U if and only if $\beta \in [-2,\ 2]$. From (13) we get

$$f(z) = \frac{1}{2\alpha - 1}\left[\ (1-z)^{-2\alpha+1} - 1\ \right]\ , \qquad \alpha \neq 1/2$$

and then the function f is not univalent if $\alpha < -1/2$ or $\alpha > 3/2$.

References

[1] A.W.Goodman, *Univalent functions* , Mariner Publishing Company Inc., 1984.

[2] M.A.Nasr, M.K.Aouf, *Caracterisation for convex and starlike functions of complex order in U* , Bull. Fac. Sci. Assiut Univ., A11(1982),nr.2, 117-121.

[3] H.Ovesea, N.N.Pascu, I.Radomir, *On the univalence of Nasr's function*, Sem.geom. function theory, Preprint 2(1991), Braşov, 73-76.

[4] H. Ovesea, *An extension of Lewandowski's univalence criteria*, Demonstratio Mathematica, XXXIX, 4(1996),699-706.

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