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ON CERTAIN CLASSES OF FUNCTIONS DEFINED BY CONVOLUTIONS

H. M. HOSSEN AND M. K. AOUF

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

Abstract. The object of the present paper is to obtain several interesting results involving coefficient estimates for analytic normalized functions belonging to certain classes defined in terms of the convolution with the extremal function for the class of starlike functions of order α, $0 \leq \alpha < 1$.

1. Introduction

Let A_1 denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. And let S denote the subclass of A_1 consisting of analytic and univalent functions $f(z)$ in the unit disc U. A function $f(z)$ of S is said to be starlike of order α if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$. We denote the class of all starlike functions of order α by $S^*(\alpha)$.

Now, the function

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \tag{1.3}$$

is the well-known extremal function for $S^*(\alpha)$. Setting

$$C(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \quad (n = 2, 3, \dots), \tag{1.4}$$

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$S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n. \tag{1.5}$$

Note that $C(\alpha, n)$ is a decreasing function of α, $0 \leq \alpha < 1$, and satisfies

$$\lim_{n\to\infty} C(\alpha, n) = \begin{cases} \infty, & \alpha < 1/2 \\ 1, & \alpha = 1/2 \\ 0, & \alpha > 1/2. \end{cases} \tag{1.6}$$

Let $(f * g)(z)$ denote the convolution or Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.7}$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.8}$$

Recently, many classes defined by convolution of $f(z)$ and $S_\alpha(z)$ have been studied by Ahuja and Silverman [1], Owa and Ahuja [11, 12], Sheil-Small, Silverman, and Silvia [15], Silverman and Silvia [16], and Ruscheweyh and Singh [14].

We denote by $P_\alpha(\beta, \gamma, A, B)$ the class of functions $f(z)$ in A_1 that satisfy the condition

$$(f * S_\alpha)'(z) \prec \frac{1 + [B + (A - B)(1 - \beta)]\gamma z}{1 + B\gamma z} \quad (z \in U) \tag{1.9}$$

for some α $(0 \leq \alpha < 1)$, β $(0 \leq \beta < 1$, γ $(0 < \gamma \leq 1)$, and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, where $\prec$ means subordination. For $f \in P_\alpha(\beta, \gamma, A, B)$, the values of $(f * S_\alpha)'(z)$ lie in a disc centered at $\dfrac{1 - [B + (A - B)(1 - \beta)]B\gamma^2}{1 - B^2\gamma^2}$ whose radius is $\dfrac{(B - A)\gamma(1 - \beta)}{1 - B^2\gamma^2}$.

We observe that, by specializing the parameters α, β, γ, A and B, we obtain the following subclasses studied by various authors:

(1) $P_{1/2}(0, 1, -1, 1) = \{f \in A_1 :\ \text{Re } f'(z) > 0,\ z \in U\}$ (Mac-Gregor [8]).

(2) $P_{1/2}(0, \gamma, -1, 1) = \left\{ f \in A_1 :\ f'(z) \prec \dfrac{1 - \gamma z}{1 + \gamma z},\ z \in U \right\}$ (Padmanabhan [13] and Caplinger and Causey [4]).

(3) $P_{1/2}(\beta, \gamma, -1, 1) = \left\{ f \in A_1 : f'(z) \prec \dfrac{1 + (2\beta - 1)\gamma z}{1 + \gamma z},\ z \in U \right\}$ (Juneja and Mogra [7]).

(4) $P_{1/2}(0,1,A,B) = \left\{ f \in A_1 : \ f'(z) \prec \dfrac{1+Az}{1+Bz}, \ z \in U \right\}$ (Mehrok [9]).

(5) $P_{1/2}(\beta,\gamma,A,B) = \left\{ f \in A_1 : \ f'(z) \prec \frac{1+[B+(A-B)(1-\beta)]\gamma z}{1+B\gamma z}, \ z \in U \right\}$ (Aouf and Owa [3]).

(6) $P_\alpha(\beta,\gamma,-1,1) = \left\{ f \in A_1 : \ (f * S_\alpha)'(z) \prec \dfrac{1+(2\beta-1)\gamma z}{1+\gamma z}, \ z \in U \right\}$ (Owa and Ahuja [12]).

(7) $P_\alpha(0,1,-1,1) = \{ f \in A_1 : \ \text{Re } (f * S_\alpha)'(z) > 0, \ z \in U \}$ (Ahuja and Owa [2]).

It is well-known that the functions in $P_{1/2}(0,1,-1,0)$ and $P_{1/2}(0,1,A,B)$ are univalent in U.

Further, we say that a function $f(z)$ in A_1 belongs to the class $Q_\alpha(\beta,\gamma,A,B)$ if and only if $zf'(z) \in P_\alpha(\beta,\gamma,A,B)$ for all $z \in U$. Finally, denote by $R_\alpha(\beta,\gamma,A,B)$ the class of functions $f(z)$ in A_1 that satisfy the condition

$$\frac{1}{z}(f * S_\alpha)(z) \prec \frac{1+[B+(A-B)(1-\beta)]\gamma z}{1+B\gamma z} \tag{1.10}$$

for some α, β, γ, A, and B as defined above. Note that

$$R_{1/2}(0,1,A,B) = \left\{ f \in A_1 : \ \frac{f(z)}{z} \prec \frac{1+Az}{1+Bz}, \ z \in U \right\}.$$

In Section 2, we first prove that

$$Q_\alpha(\beta,\gamma,A,B) \subset P_\alpha(\beta,\gamma,A,B) \subset R_\alpha(\beta,\gamma,A,B),$$

and we then determine coefficient inequalities for the functions belonging to these classes. Finally, the coefficient inequalities for some subclasses of $P_\alpha(\beta,\gamma,A,B)$ and $Q_\alpha(\beta,\gamma,A,B)$ are obtained.

2. **Coefficient Inequalities**

First we examine the relationship between $P_\alpha(\beta,\gamma,A,B)$ and $Q_\alpha(\beta,\gamma,A,B)$. We need the following very useful result due to Jack [6], and Suffridge [17].

Lemma 1. *Let $w(z)$ be analytic in U with $w(0)=0$. If $|w(z_0)| = \max_{|z|=r} |w(z)|$, then we have*

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

Theorem 1. $Q_\alpha(\beta,\gamma,A,B) \subset P_\alpha(\beta,\gamma,A,B)$.

Proof. Let $f \in Q_\alpha(\beta,\gamma,A,B)$. Then $zf'(z) \in P_\alpha(\beta,\gamma,A,B)$ and therefore

$$(zf' * S_\alpha)'(z) \prec g(z), \tag{2.1}$$

where $g(z) = \dfrac{1+[B+(A-B)(1-\beta)]\gamma z}{1+B\gamma z}$ is convex univalent in U. In view of the principle of subordination and the Schwarz's Lemma [10], it follows that (2.1) is equivalent to

$$\left|\frac{(zf' * S_\alpha)'(z) - 1}{B\gamma(zf' * S_\alpha)'(z) - [B+(A-B)(1-\beta)]\gamma}\right| < 1. \tag{2.2}$$

Define $w(z)$ by

$$\frac{(zf' * S_\alpha)(z)}{z} = \frac{1+[B+(A-B)(1-\beta)]\gamma w(z)}{1+B\gamma w(z)}. \tag{2.3}$$

We observe that

$$\frac{(zf' * S_\alpha)(z)}{z} = (f * S_\alpha)'(z).$$

Thus (2.3) can be written as

$$w(z) = \frac{(f * S_\alpha)'(z) - 1}{[B+(A-B)(1-\beta)]\gamma - B\gamma(f * S_\alpha)'(z)}. \tag{2.4}$$

Note that $w(z)$ is analytic in U and $w(0) = 0$. We need to show that $|w(z)| < 1$ for all $z \in U$. On the contrary, suppose $|w(z)| \not< 1$. Then by Lemma 1, there exists a point $z_0 \in U$ such that $|w(z_0)| = 1$, $z_0 w'(z_0) = kw(z_0)$ for some $k \geq 1$. Therefore, (2.3) yields

$$(z_0 f' * S_\alpha)'(z_0) - 1 = \frac{(A-B)(1-\beta)\gamma w(z_0)(1+T(z_0))}{1+B\gamma w(z_0)},$$

and

$$B\gamma(z_0 f' * S_\alpha)'(z_0) - [B+(A-B)(1-\beta)]\gamma =$$
$$= \frac{(B-A)(1-\beta)\gamma[1-B\gamma w(z_0)T(z_0)]}{1+B\gamma w(z_0)}$$

where $T(z_0) = \dfrac{k}{1+B\gamma w(z_0)}$, and hence (2.2) implies that

$$\left|\frac{1+T(z_0)}{1-B\gamma w(z_0)T(z_0)}\right| < 1.$$

This last inequality gives

$$(1-B^2\gamma^2)|T(z_0)|^2 < -2\text{Re}\,[(1+B\gamma w(z_0))T(z_0)]. \tag{2.5}$$

Since the right side of (2.5) is equal to $-2k$ and $k \geq 1$, we conclude that (2.5) is not possible. This contradiction thereby shows that $|w(z)| < 1$ for all $z \in U$, and hence (2.4) immediately proves that $f \in P_\alpha(\beta, \gamma, A, B)$. The proof of the theorem is completed.

Theorem 2. $P_\alpha(\beta, \gamma, A, B) \subset R_\alpha(\beta, \gamma, A, B)$.

Proof. Let $f \in P_\alpha(\beta, \gamma, A, B)$. Then it follows that

$$\frac{1}{z}(zf' * S_\alpha)(z) \prec g(z),$$

where $g(z) = \dfrac{1 + [B + (A - B)(1 - \beta)]\gamma z}{1 + B\gamma z}$ is convex univalent in U and hence $h(z) = zf'(z) \in R_\alpha(\beta, \gamma, A, B)$. Therefore, in view of (1.10), we have

$$\frac{1}{z}\left(\int_0^z \frac{h(t)}{t} dt * S_\alpha\right)(z) = \int_0^1 \frac{(h * S_\alpha)(tz)}{tz} dt \prec g(z).$$

This implies that

$$f(z) = \int_0^z \frac{h(t)}{t} dt \in R_\alpha(\beta, \gamma, A, B),$$

which completes the proof of the theorem.

Corollary 1. *If $f(z) \in P_\alpha(\beta, \gamma, A, B)$, then we have*

$$\left|\arg \frac{1}{z}(f * S_\alpha)(z)\right| \leq \sin^{-1}\left(\frac{(B - A)\gamma(1 - \beta)|z|}{1 - B\gamma^2[B + (A - B)(1 - \beta)]|z|^2}\right).$$

The bound is sharp.

We next obtain a sufficient condition in terms of the modulus of the coefficients for a function to be in $P_\alpha(\beta, \gamma, A, B)$.

Theorem 3. *Let the function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty} n(1 + B\gamma)C(\alpha, n)|a_n| \leq (B - A)\gamma(1 - \beta) \tag{2.6}$$

for $0 \leq \alpha < 1$, $0 \leq \beta < 1$, $0 < \gamma \leq 1$, and $-1 \leq A < B \leq 1$, $0 < B \leq 1$. Then $f(z)$ is in the class $P_\alpha(\beta, \gamma, A, B)$.

Proof. We use a method of Clunie and Keogh [5]. Assuming the inequality (2.6), we have

$$|(f * S_\alpha)'(z) - 1| - \gamma|B(f * S_\alpha)'(z) - [B + (A - B)(1 - \beta)]| =$$

$$= \left|\sum_{n=2}^{\infty} nC(\alpha, n)a_n z^{n-1}\right| - \gamma\left|(B - A)(1 - \beta) + \sum_{n=2}^{\infty} BnC(\alpha, n)a_n z^{n-1}\right| \leq$$

$$\leq \sum_{n=2}^{\infty} nC(\alpha,n)|a_n||z|^{n-1} - \gamma\left\{(B-A)(1-\beta) - \sum_{n=2}^{\infty} BnC(\alpha,n)|a_n||z|^{n-1}\right\} \leq$$

$$\leq \sum_{n=2}^{\infty} n(1+B\gamma)C(\alpha,n)|a_n| - (B-A)\gamma(1-\beta) \leq 0$$

for all $z \in U$. Consequently, by the maximum modulus theorem, it follows that $f(z) \in P_\alpha(\beta,\gamma,A,B)$. The equality in (2.6) is attained for the functions of the form

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n \quad (n \geq 2).$$

Example. The function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ given by

$$(f * S_\alpha)(z) = z + \sum_{n=2}^{\infty} C(\alpha,n) a_n z^n =$$

$$= -\frac{[B + (A-B)(1-\beta)]}{B} z + \frac{(B-A)(1-\beta)}{B^2\gamma} \ln(1 - B\gamma z) \qquad (2.7)$$

belongs to $P_\alpha(\beta,\gamma,A,B)$ but

$$\sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)}|a_n| = \sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)} \frac{(B-A)(1-\beta)}{nC(\alpha,n)} B^{n-2}\gamma^{n-1} =$$

$$= \sum_{n=2}^{\infty} (1+B\gamma)(B\gamma)^{n-2} > 1$$

for each α,β,γ,A,B $(0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,\ -1 \leq A < B \leq 1,\ 0 < B \leq 1)$. This example shows that the converse of Theorem 3 may not be true.

Motivated by Theorem 3 and the above Example, we now consider a class $H_\alpha(\beta,\gamma,A,B)$ of precisely those functions in A_1 which are characterized by the condition (2.6): that is, $f(z) \in H_\alpha(\beta,\gamma,A,B)$ if and only if $f(z)$ satisfies (2.6) for some α,β,γ,A,B $(0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1,\ -1 \leq A < B \leq 1,\ 0 < B \leq 1)$. Clearly, $H_\alpha(\beta,\gamma,A,B) \subset P_\alpha(\beta,\gamma,A,B)$. This containment is proper because $f(z)$ given by (2.7) belongs to $P_\alpha(\beta,\gamma,A,B) - H_\alpha(\beta,\gamma,A,B)$. We next prove a theorem about convolutions of functions in $H_\alpha(\beta,\gamma,A,B)$. But first we need the following

Lemma 2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H_\alpha(\beta,\gamma,A,B)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A_1$ with $|b_n| \leq 1$ for every n, then $(f * g)(z) \in H_\alpha(\beta,\gamma,A,B)$.*

Proof. The result follows from (2.6) upon noting that

$$\sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)}|a_n||b_n| \leq \sum_{n=2}^{\infty} \frac{n(1+B\gamma)C(\alpha,n)}{(B-A)\gamma(1-\beta)}|a_n| \leq 1.$$

Remark. The condition $|b_n| \leq 1$ is best possible because if $|b_n| > 1$ for some n, then

$$\left(z + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n\right) * g(z) \notin H_\alpha(\beta,\gamma,A,B).$$

Theorem 4. *If $f, g \in H_\alpha(\beta,\gamma,A,B)$ with*

$$\alpha \leq \frac{1+B\gamma\beta}{1+B\gamma} \tag{2.8}$$

then $f * g(z) \in H_\alpha(\beta,\gamma,A,B)$.

Proof. According to Lemma 2, it suffices to show that the modulus of the n-th coefficient, $|b_n|$, is bounded above by 1. Note that

$$C(\alpha,n) = \frac{\prod_{k=2}^{\infty}(k-2\alpha)}{(n-1)!} \geq \frac{2(1-\alpha)}{(n-1)!}\prod_{k=3}^{n}(k-2) > \frac{(B-A)(1-\alpha)}{Bn}.$$

Thus from (2.6) we have

$$|b_n| \leq \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} <$$

$$< \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)} \frac{Bn}{(B-A)(1-\alpha)} = \frac{B\gamma(1-\beta)}{(1-\alpha)(1+B\gamma)}. \tag{2.9}$$

This last expression is bounded above by 1 if (2.8) holds and the proof is completed.

Remark. The condition (2.8) cannot be eliminated. The function

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)} z^n = z + a_n z^n \quad (n \geq 2)$$

is in $H_\alpha(\beta,\gamma,A,B)$ but $f_n * f_n(z) \notin H_\alpha(\beta,\gamma,A,B)$ for α close enough to 1 to assure that $a_n > 1$.

With the aid of Theorem 3, we have

Theorem 5. *Let the function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty} n^2(1+B\gamma)C(\alpha,n)|a_n| \leq (B-A)\gamma(1-\beta) \tag{2.10}$$

for $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1$ and $-1 \leq A < B \leq 1,\ 0 < B \leq 1$. Then $f(z)$ is in the class $Q_\alpha(\beta,\gamma,A,B)$.

Proof. We note that $f(z) \in Q_\alpha(\beta,\gamma,A,B)$ if and only if $zf'(z) \in P_\alpha(\beta,\gamma,A,B)$. Since $zf'(z) = z + \sum_{n=2}^{\infty} na_n z^n$, we may replace a_n by na_n in Theorem 3. Further, the equality in (2.10) holds for the functions of the form

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{n^2(1+B\gamma)C(\alpha,n)} z^n \quad (n \geq 2). \tag{2.11}$$

Following the method of Theorem 3, we obtain a sufficient condition in terms of the modulus of the coefficients for a function to be in $R_\alpha(\beta,\gamma,A,B)$.

Theorem 6. *Let the function $f(z)$ defined by (1.1) satisfies the condition*

$$\sum_{n=2}^{\infty}(1+B\gamma)C(\alpha,n)|a_n| \leq (B-A)\gamma(1-\beta) \tag{2.12}$$

for $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 < \gamma \leq 1$ and $-1 \leq A < B \leq 1,\ 0 < B \leq 1$. Then $f(z)$ is in the class $R_\alpha(\beta,\gamma,A,B)$. The equality in (2.12) is attained for the functions of the form

$$f_n(z) = z + \frac{(B-A)\gamma(1-\beta)}{(1+B\gamma)C(\alpha,n)} z^n \quad (n \geq 2). \tag{2.13}$$

Remark. The proof of Theorem 6 is omitted. Furthermore, analogous to subclass $H_\alpha(\beta,\gamma,A,B)$ of $P_\alpha(\beta,\gamma,A,B)$ and Theorem 4, it is a simple exercise to introduce and study corresponding subclasses of $Q_\alpha(\beta,\gamma,A,B)$ and $R_\alpha(\beta,\gamma,A,B)$.

The next theorem gives the coefficient bounds for the functions in the class $P_\alpha(\beta,\gamma,A,B)$.

Theorem 7. *Let the function $f(z)$ defined by (1.1) be in the class $P_\alpha(\beta,\gamma,A,B)$. Then we have*

$$|a_n| \leq \frac{(B-A)\gamma(1-\beta)}{nC(\alpha,n)} \quad (n \geq 2). \tag{2.14}$$

These bounds are sharp.

Proof. Let $f(z) \in P_\alpha(\beta,\gamma,A,B)$. Then it follows from the definition of subordination

$$(f * S_\alpha)'(z) = \frac{1+[B+(A-B)(1-\beta)]\gamma w(z)}{1+B\gamma w(z)}, \tag{2.15}$$

where $w(z) = \sum_{k=1}^{\infty} t_k z^k$ is analytic and satisfies the condition $|w(z)| < 1$ for all z in U. On simplification, (2.15) gives

$$\gamma \left[(B-A)(1-\beta) + \sum_{n=2}^{\infty} BnC(\alpha,n)a_n z^{n-1} \right] \left[\sum_{n=1}^{\infty} t_n z^n \right] =$$

$$= -\sum_{n=2}^{\infty} nC(\alpha,n)a_n z^{n-1}. \tag{2.16}$$

Equating corresponding coefficients on both sides of (2.16) we find that the coefficient a_n on the right side depends only on the coefficients $a_2, a_3, \ldots, a_{n-1}$ on the left side. Therefore, since $|w(z)| < 1$, (2.16) gives

$$\gamma \left| (B-A)(1-\beta) + \sum_{k=2}^{n-1} BkC(\alpha,k)a_k z^{k-1} \right| \geq \left| \sum_{k=2}^{n} kC(\alpha,k)a_k z^{k-1} - \sum_{k=n+1}^{\infty} b_k z^{k-1} \right|$$

for all $n \geq 2$. Writting $z = re^{i\theta}$, $r < 1$, squaring both sides of the preceeding inequality and then integrating, we obtain

$$\gamma^2 \left[(B-A)^2(1-\beta)^2 + \sum_{k=2}^{n-1} B^2k^2(C(\alpha,k))^2|a_k|^2 r^{2(k-1)} \right] \geq$$

$$\geq \sum_{k=2}^{n} k^2(C(\alpha,k))^2|a_k|^2 r^{2(k-1)} + \sum_{k=n+1}^{\infty} |b_k|^2 r^{2(k-1)}.$$

Taking the limit as $r \to 1^-$, we have

$$\gamma^2 \left[(B-A)^2(1-\beta)^2 + \sum_{k=2}^{n-1} B^2k^2(C(\alpha,k))^2|a_k|^2 \right] \geq$$

$$\geq n^2(C(\alpha,n))^2|a_n|^2 + \sum_{k=2}^{n-1} k^2(C(\alpha,k))^2|a_k|^2. \tag{2.17}$$

Since $0 < \gamma \leq 1$ and $0 < B \leq 1$, (2.17) immediately yields

$$(B-A)^2\gamma^2(1-\beta)^2 \geq n^2(C(\alpha,n))^2|a_n|^2$$

which proves (2.14). The bounds in (2.14) are sharp for the functions $f(z)$ defined by

$$(f * S_\alpha)(z) = \int_0^z \frac{1-[B+(A-B)(1-\beta)]\gamma t^{n-1}}{1-B\gamma t^{n-1}} dt \tag{2.18}$$

for $n \geq 2$ and for all $z \in U$.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $Q_\alpha(\beta, \gamma, A, B)$. Then we have*

$$|a_n| \le \frac{(B-A)\gamma(1-\beta)}{n^2 C(\alpha, n)} \quad (n \ge 2). \tag{2.19}$$

These bounds are sharp.

Proof. We need only replace a_n by na_n in Theorem 7.

Remark. We can show that the inclusion $Q_\alpha(\beta, \gamma, A, B) \subset P_\alpha(\beta, \gamma, A, B)$ and $H_\alpha(\beta, \gamma, A, B) \subset P_\alpha(\beta, \gamma, A, B)$ are both proper. In particular, for $f(z)$ given by (2.18) it follows that

$$f(z) = z + \frac{(B-A)\gamma(1-\beta)}{nC(\alpha, n)} z^n + \dots = z + a_n z^n + \dots$$

is in $P_\alpha(\beta, \gamma, A, B)$ but $f \notin Q_\alpha(\beta, \gamma, A, B)$ and $f \notin H_\alpha(\beta, \gamma, A, B)$ because a_n exceeds the coefficients bounds of the above Corollary 2 and (2.6).

By using the arguments similar to Theorem 7, we obtain the following

Theorem 8. *Let the function $f(z)$ defined by (1.1) be in the class $R_\alpha(\beta, \gamma, A, B)$. Then we have*

$$|a_n| \le \frac{(B-A)\gamma(1-\beta)}{C(\alpha, n)} \quad (n \ge 2). \tag{2.20}$$

These bounds are sharp for the function $f(z)$ given by

$$(f * S_\alpha)(z) = \left(\frac{1 - [B + (A-B)(1-\beta)]\gamma z^{n-1}}{1 - B\gamma z^{n-1}} \right) z. \tag{2.21}$$

3. **Subclasses of $P_\alpha(\beta, \gamma, A, B)$ and $Q_\alpha(\beta, \gamma, A, B)$**

In view of Theorem 3, we introduce the following classes. Let $P_\alpha(\beta, \gamma, A, B; k)$ be the subclasses of $P_\alpha(\beta, \gamma, A, B)$ consisting of functions of the form

$$f(z) = z + \sum_{i=1}^{k} B_i p_i z^i + \sum_{n=k+1}^{\infty} a_n z^n, \tag{3.1}$$

where $0 \le p_i < 1$, $0 \le \sum_{i=2}^{k} p_i < 1$, and

$$B_i = \frac{(B-A)\gamma(1-\beta)}{i(1+B\gamma)C(\alpha, i)} \quad (i = 2, 3, \dots, k). \tag{3.2}$$

Further, let $Q_\alpha(\beta,\gamma,A,B;k)$ be the subclass of $Q_\alpha(\beta,\gamma,A,B)$ consisting of functions of the form

$$f(z)=z+\sum_{i=2}^{k}E_ip_iz^i+\sum_{n=k+1}^{\infty}a_nz^n, \tag{3.3}$$

where $0\le p_i<1;\ 0\le\sum_{i=2}^{k}p_i<1$, and

$$E_i=\frac{(B-A)\gamma(1-\beta)}{i^2(1+B\gamma)C(\alpha,i)}\quad (i=2,3,\dots,k). \tag{3.4}$$

Theorem 9. *Let the function $f(z)$ defined by (3.1) satisfies the condition*

$$\sum_{n=k+1}^{\infty}n(1+B\gamma)C(\alpha,n)|a_n|\le(B-A)\gamma(1-\beta)\left(1-\sum_{i=2}^{k}p_i\right) \tag{3.5}$$

for $0\le\alpha<1,\ 0\le\beta<1,\ 0<\gamma\le1$, and $-1\le A<B\le1,\ 0<B\le1$. Then $f(z)$ is in the class $P_\alpha(\beta,\gamma,A,B;k)$.

Proof. By virtue of Theorem 3, we note that

$$f(z)=z+\sum_{i=2}^{k}B_ip_iz^i+\sum_{n=k+1}^{\infty}a_nz^n$$

belongs to the class $P_\alpha(\beta,\gamma,A,B;k)$ if

$$\sum_{i=2}^{k}i(1+B\gamma)C(\alpha,i)B_ip_i+\sum_{n=k+1}^{\infty}n(1+B\gamma)C(\alpha,n)|a_n|\le(B-A)\gamma(1-\beta), \tag{3.6}$$

or if

$$\sum_{i=2}^{k}(B-A)\gamma(1-\beta)p_i+\sum_{n=k+1}^{\infty}n(1+B\gamma)C(\alpha,n)|a_n|\le(B-A)\gamma(1-\beta). \tag{3.7}$$

This is equivalent to the condition (3.5). Further, by taking the function given by

$$f(z)=z+\sum_{i=2}^{k}B_ip_iz^i+\frac{(B-A)\gamma(1-\beta)}{n(1+B\gamma)C(\alpha,n)}z^n\quad (n\ge k+1), \tag{3.8}$$

we can show that the result (3.5) is sharp.

Putting $p_i=0$ $(i=2,3,\dots,k)$ in Theorem 9, we have

Corollary 3. *Let the function $f(z)$ defined by (3.1) with $p_i=0$ $(i=2,3,\dots,k)$. If f satisfies*

$$\sum_{n=k+1}^{\infty}n(1+B\gamma)C(\alpha,n)|a_n|\le(B-A)\gamma(1-\beta) \tag{3.9}$$

for $0 \le \alpha < 1,\ 0 \le \beta < 1,\ 0 < \gamma \le 1,$ *and* $-1 \le A < B \le 1,\ 0 < B \le 1,$ *then* $f(z) \in P_\alpha(\beta, \gamma, A, B; k)$.

Similarly, we obtain

Theorem 10. *Let the function* $f(z)$ *defined by (3.2) satisfies the condition*

$$\sum_{n=k+1}^{\infty} n^2(1+B\gamma)C(\alpha,n)|a_n| \le (B-A)\gamma(1-\beta)\left(1-\sum_{i=2}^{k} p_i\right) \tag{3.10}$$

for $0 \le \alpha < 1,\ 0 \le \beta < 1,\ 0 < \gamma \le 1,$ *and* $-1 \le A < B \le 1,\ 0 < B \le 1.$ *Then* $f(z)$ *is in the class* $Q_\alpha(\beta, \gamma, A, B; k)$.

Corollary 4. *Let the function* $f(z)$ *be defined by (3.2) with* $p_i = 0$ $(i = 2, 3, \ldots, k)$. *If* $f(z)$ *satisfies*

$$\sum_{n=k+1}^{\infty} n^2(1+B\gamma)C(\alpha,n)|a_n| \le (B-A)\gamma(1-\beta) \tag{3.11}$$

for $0 \le \alpha < 1,\ 0 \le \beta < 1,\ 0 < \gamma \le 1,$ *and* $-1 \le A < B \le 1,\ 0 < B \le 1,$ *then* $f(z) \in Q_\alpha(\beta, \gamma, A, B; k)$.

Remark. Putting $A = -1$ and $B = 1$ in the above theorems we get the results obtained by Ahuja and Owa [2].

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Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura, Egypt
E-mail address: sinfac@mum.mans.eun.eg