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GENERALIZATION OF CERTAIN CLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Dedicated to Professor Petru T. Mocanu on his 70^{th} birthday

Abstract. The object of the present paper is to obtain coefficient estimates, some properties, distortion theorem and closure theorems for the classes $R_n^*(\alpha)$ of analytic and univalent functions with negative coefficients, defined by using the n-th order Ruscheweyh derivative. We also obtain several interesting results for the modified Hadamard product of functions belonging to the classes $R_n^*(\alpha)$. Further, we obtain radii of close-to-convexity, starlikeness and convexity and integral operators for the classes $R_n^*(\alpha)$.

1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. We denote by S the subclass of univalent functions $f(z)$ in A. The Hadamard product of two functions $f(z) \in A$ and $g(z) \in A$ will be denoted by $f * g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

then

$$f * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

Let

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} \tag{1.4}$$

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for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $z \in U$, where $\mathbb{N} = \{1, 2, 3, \dots\}$. This symbol $D^n f(z)$ was named the n-th order Ruscheweyh derivative of $f(z)$ by Al-Amiri [3]. We note that $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$. By using the Hadamard product, Ruscheweyh [5] observed that if

$$D^\beta f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z) \quad (\beta \geq -1) \tag{1.5}$$

then (1.4) is equivalent to (1.5) when $\beta = n \in \mathbb{N}_0$.

It is easy to see that

$$D^n f(z) = k + \sum_{k=2}^{\infty} \delta(n,k) a_k z^k, \tag{1.6}$$

where

$$\delta(n,k) = \binom{n+k-1}{n}. \tag{1.7}$$

Note that

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z) \quad (\text{cf. [5]}). \tag{1.8}$$

Let $R_n(\alpha)$ denote the classes of functions $f(z) \in A$ which satisfy the condition

$$\text{Re}\left\{\frac{z(D^n f(z))'}{D^n f(z)}\right\} > \alpha, \quad (z \in U) \tag{1.9}$$

for some α $(0 \leq \alpha < 1)$ and $n \in \mathbb{N}_0$. The class $R_n(\alpha)$ was studied by Ahuja [1,2].

From (1.8) and (1.9) it follows that a function $f(z)$ in A belongs to $R_n(\alpha)$ is and only if

$$\text{Re}\left\{\frac{D^{n+1} f(z)}{D^n f(z)}\right\} > \frac{n+\alpha}{n+1} \quad (z \in U). \tag{1.10}$$

Let T denote the subclass of S consisting of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{1.11}$$

In the present paper we introduce the following classes $R_n^*(\alpha)$ by using the n-th order Ruscheweyh derivative of $f(z)$, defined as follows:

Definition. We say that $f(z)$ is in the class $R_n^*(\alpha)$ $(0 \leq \alpha < 1,\ n \in \mathbb{N}_0)$, if $f(z)$ defined by (1.11) satisfies the condition (1.10).

We note that $R_n^*(0) = R_n^*$ was studied by Owa [4] and $R_0^*(\alpha) = T^*(\alpha)$ (the class of starlike functions of order α) and $R_1^*(\alpha) = C(\alpha)$ (the class of convex functions of order α), were studied by Silverman [7]. Hence $R_n^*(\alpha)$ is a subclass of $T^*(\alpha) \subset S$. Further, we can show that $R_{n+1}^*(\alpha) \subset R_n^*(\alpha)$ for every $n \in \mathbb{N}_0$.

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.11). Then $f(z)$ is in the class $R_n^*(\alpha)$ if and only if*

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \leq 1-\alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds and let $|z|=1$. Then we get

$$\left|\frac{D^{n+1}f(z)}{D^n f(z)}-1\right| = \left|\frac{-\sum_{k=2}^{\infty}(\delta(n+1,k)-\delta(n,k))a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k z^{k-1}}\right| \leq$$

$$\leq \frac{\sum_{k=2}^{\infty}\left(\frac{k-1}{n+1}\right)\delta(n,k)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k|z|^{k-1}} \leq \frac{\sum_{k=2}^{\infty}\left(\frac{k-1}{n+1}\right)\delta(n,k)a_k}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k} \leq \frac{1-\alpha}{n+1}.$$

This shows that the values of $\frac{D^{n+1}f(z)}{D^n f(z)}$ lies in a circle centered at $w=1$ whose radius is $\frac{1-\alpha}{n+1}$. Hence $f(z)$ satisfies the condition (1.10) hence further, $f(z) \in R_n^*(\alpha)$.

For the converse, assume that the function $f(z)$ defined by (1.11) belongs to the class $R_n^*(\alpha)$. Then we have

$$\text{Re}\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \text{Re}\left\{\frac{1-\sum_{k=2}^{\infty}\delta(n+1,k)a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\delta(n,k)a_k z^{k-1}}\right\} > \frac{n+\alpha}{n+1} \tag{2.2}$$

for $0 \leq \alpha < 1$ and $z \in U$. Choose values of z on the real axis so that $\frac{D^{n+1}f(z)}{D^n f(z)}$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we get

$$(n+1)\left(1-\sum_{k=2}^{\infty}\delta(n+1,k)a_k\right) \geq (n+\alpha)\left(1-\sum_{k=2}^{\infty}\delta(n,k)a_k\right) \tag{2.3}$$

which gives (2.1). Finally the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(k-\alpha)\delta(n,k)} z^k \quad (k \geq 2) \tag{2.4}$$

is an extremal function for the theorem.

Corollary 1. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$. Then*

$$a_k \leq \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \quad (k \geq 2). \tag{2.5}$$

The equality in (2.5) is attained for the function $f(z)$ given by (2.4).

3. Some properties of the class $R_n^*(\alpha)$

Theorem 2. *Let $0 \leq \alpha_1 \leq \alpha_2 < 1$ and $n \in \mathbb{N}_0$. Then we have*

$$R_n^*(\alpha_1) \supseteq R_n^*(\alpha_2). \tag{3.1}$$

Proof. Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha_2)$ and $\alpha_1 = \alpha_2 - \varepsilon$. Then, by Theorem 1, we have

$$\sum_{k=2}^{\infty} (k-\alpha_2)\delta(n,k)a_k \leq 1-\alpha_2$$

and

$$\sum_{k=2}^{\infty} \delta(n,k)a_k \leq \frac{1-\alpha_2}{2-\alpha_2} < 1. \tag{3.2}$$

Consequently

$$\sum_{k=2}^{\infty} (k-\alpha_1)\delta(n,k)a_k = \sum_{k=2}^{\infty} (k-\alpha_2)\delta(n,k)a_k + \varepsilon \sum_{k=2}^{\infty} \delta(n,k)a_k \leq 1-\alpha_1. \tag{3.3}$$

This completes the proof of Theorem 2 with the aid of Theorem 1.

Theorem 3. $R_{n+1}^*(\alpha) \subseteq R_n^*(\alpha)$ *for* $0 \leq \alpha < 1$ *and* $n \in \mathbb{N}_0$.

Proof. Let the function $f(z)$ defined by (1.11) be in the class $R_{n+1}^*(\alpha)$; then

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n+1,k)a_k \leq 1-\alpha \tag{3.4}$$

and since

$$\delta(n,k) \leq \delta(n+1,k) \text{ for } k \geq 2, \tag{3.5}$$

we have

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \leq \sum_{k=2}^{\infty}(k-\alpha)\delta(n+1,k)a_k \leq 1-\alpha. \tag{3.6}$$

The result follows from Theorem 1.

4. **Distortion theorem**

Theorem 4. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$. Then we have for $|z| = r < 1$*

$$r - \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \leq |f(z)| \leq r + \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \tag{4.1}$$

and

$$1 - \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r. \tag{4.2}$$

The result is sharp.

Proof. In view of Theorem 1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{1-\alpha}{(2-\alpha)(n+1)}. \tag{4.3}$$

Consequently, we have

$$|f(z)| \geq r - r^2\sum_{k=2}^{\infty} a_k \geq r - \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \tag{4.4}$$

and

$$|f(z)| \leq r + r^2\sum_{k=2}^{\infty} a_k \leq r + \frac{1-\alpha}{(2-\alpha)(n+1)}r^2 \tag{4.5}$$

which prove the assertion (4.1).

From (4.3) and Theorem 1, it follows also that

$$\sum_{k=2}^{\infty} ka_k \leq \frac{1-\alpha}{n+1} + \alpha\sum_{k=2}^{\infty} a_k \leq \frac{2(1-\alpha)}{(2-\alpha)(n+1)}. \tag{4.6}$$

Consequently, we have

$$|f'(z)| \geq 1 - r\sum_{k=2}^{\infty} ka_k \geq 1 - \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r \tag{4.7}$$

and

$$|f'(z)| \leq 1 + r\sum_{k=2}^{\infty} ka_k \leq 1 + \frac{2(1-\alpha)}{(2-\alpha)(n+1)}r, \tag{4.8}$$

which prove the assertion (4.2). This completes the proof of Theorem 4.

The bounds in (4.1) and (4.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \quad (z = \pm r). \tag{4.9}$$

Corollary 2. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$. Then the unit disc U is mapped onto a domain that contains the disc*

$$|w| < \frac{(2-\alpha)(n+1) - (1-\alpha)}{(2-\alpha)(n+1)} \tag{4.10}$$

The result is sharp with extremal function $f(z)$ given by (4.9).

5. **Closure theorems**

Let the functions $f_i(z)$ be defined, for $i = 1, 2, \ldots, m$, by

$$f_i(z) = z - \sum_{k=2}^{\infty} a_{k,i} z^k \quad (a_{k,i} \geq 0,\ k \geq 2) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the classes $R_n^*(\alpha)$.

Theorem 5. *Let the functions $f_i(z)$ defined by (5.1) be in the class $R_n^*(\alpha)$ for every $i = 1, 2, \ldots, m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \quad (c_i \geq 0) \tag{5.2}$$

is also in the class $R_n^*(\alpha)$, where

$$\sum_{i=1}^{m} c_i = 1. \tag{5.3}$$

Proof. According to the definition of $h(z)$, we can write

$$h(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{m} c_i a_{k,i} \right) z^k. \tag{5.4}$$

Further, since $f_i(z)$ are in $R_n^*(\alpha)$ for every $i = 1, 2, \ldots, m$, we get

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) a_{k,i} \leq 1 - \alpha \tag{5.5}$$

for every $i = 1, 2, \ldots, m$. Hence we can see that

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) \left(\sum_{i=1}^{m} c_i a_{k,i} \right) = \sum_{i=1}^{m} c_i \left(\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) a_{k,i} \right) \leq$$

$$= \left(\sum_{i=1}^{m} c_i \right) (1-\alpha) \leq 1-\alpha \tag{5.6}$$

with the aid of (5.5). This proves that the function $h(z)$ is in the class $R_n^*(\alpha)$ by means of Theorem 1. Thus we have the theorem.

Theorem 6. *The class $R_n^*(\alpha)$ is closed under convex linear combinations.*

Proof. Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (5.1) be in the class $R_n^*(\alpha)$. Then it is sufficient to prove that the function

$$h(z) = \lambda f_1(z) + (1-\lambda) f_2(z) \quad (0 \leq \lambda \leq 1) \tag{5.7}$$

is in the class $R_n^*(\alpha)$. Since, for $0 \leq \lambda \leq 1$,

$$h(z) = z - \sum_{k=2}^{\infty} \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} z^k, \tag{5.8}$$

we readily have

$$\sum_{k=2}^{\infty} (k-\alpha)\delta(n,k) \{\lambda a_{k,1} + (1-\lambda) a_{k,2}\} \leq 1-\alpha, \tag{5.9}$$

by means of Theorem 1, which implies that $h(z) \in R_n^*(\alpha)$.

Theorem 7. *Let*

$$f_1(z) = z \tag{5.10}$$

and

$$f_k(z) = z - \frac{1-\alpha}{(k-\alpha)\delta(n,k)} z^k \quad (k \geq 2) \tag{5.11}$$

for $0 \leq \alpha < 1$ and $n \in \mathbb{N}_0$. Then $f(z)$ is in the class $R_n^*(\alpha)$ if and only if can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) \tag{5.12}$$

where $\lambda_k \geq 0$ and

$$\sum_{k=1}^{\infty} \lambda_k = 1. \tag{5.13}$$

Proof. Assume that

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \lambda_k z^k. \tag{5.14}$$

Then we have

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \cdot \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \lambda_k = \sum_{k=2}^{\infty} \lambda_k = 1 - \lambda_1 \leq 1. \tag{5.15}$$

So by Theorem 1, $f(z) \in R_n^*(\alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.11) belongs to the class $R_n^*(\alpha)$. Again, with the aid of Theorem 1, we get

$$a_k \leq \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \quad (k \geq 2). \tag{5.16}$$

Setting

$$\lambda_k = \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_k \quad (k \geq 2), \tag{5.17}$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k. \tag{5.18}$$

Hence, we can see that $f(z)$ can be expressed in the form (5.12). This completes the proof of Theorem 7.

Corollary 3. *The extreme points of the class $R_n^*(\alpha)$ are the functions $f_1(z)$ and $f_k(z)$ $(k \geq 2)$ given by Theorem 7.*

6. **Modified Hadamard product**

Let the functions $f_i(z)$ $(i = 1, 2)$ be defined (5.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k. \tag{6.1}$$

Theorem 8. *Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (5.1) be in the class $R_n^*(\alpha)$. Then $f_1 * f_2(z) \in R_n^*(\beta(n,\alpha))$, where*

$$\beta(n,\alpha) = \frac{(n+1) - 2\left(\dfrac{1-\alpha}{2-\alpha}\right)^2}{(n+1) - \left(\dfrac{1-\alpha}{2-\alpha}\right)^2}. \tag{6.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest $\beta = \beta(n,\alpha)$ such that

$$\sum_{k=2}^{\infty} \frac{(k-\beta)\delta(n,k)}{1-\beta} a_{k,1}a_{k,2} \leq 1. \tag{6.3}$$

Since

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_{k,1} \leq 1 \tag{6.4}$$

and

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_{k,2} \leq 1, \tag{6.5}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{6.6}$$

Thus it is sufficient to show that

$$\frac{(k-\beta)\delta(n,k)}{1-\beta} a_{k,1}a_{k,2} \leq \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \sqrt{a_{k,1}a_{k,2}} \quad (k \geq 2), \tag{6.7}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq 2). \tag{6.8}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1-\alpha}{(k-\alpha)\delta(n,k)} \quad (k \geq 2). \tag{6.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{(k-\alpha)\delta(n,k)} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq 2), \tag{6.10}$$

or, equivalently, that

$$\beta \leq \frac{\delta(n,k) - k\left(\dfrac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k) - \left(\dfrac{1-\alpha}{k-\alpha}\right)^2} \quad (k \geq 2). \tag{6.11}$$

Since

$$A(k) = \frac{\delta(n,k) - k\left(\dfrac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k) - \left(\dfrac{1-\alpha}{k-\alpha}\right)^2} \tag{6.12}$$

is an increasing function of k $(k \geq 2)$, letting $k = 2$ in (6.12), we obtain

$$\beta \leq A(2) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)^2}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)^2}, \tag{6.13}$$

which completes the proof of the theorem. Finally, by taking the functions $f_i(z)$ given by

$$f_i(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \quad (i = 1, 2), \tag{6.14}$$

we can see that the result is sharp.

Corollary 4. *For $f_1(z)$ and $f_2(z)$ as in Theorem 8, we have*

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1} a_{k,2}} z^k \tag{6.15}$$

belongs to the class $R_n^*(\alpha)$.

The result follows from the inequality (6.6). It is sharp for the same functions $f_i(z)$ $(i = 1, 2)$ as in Theorem 8.

Theorem 9. *Let $f_1(z) \in R_n^*(\alpha)$ and $f_2(z) \in R_n^*(\beta)$, then $f_1 * f_2(z) \in R_n^*(\gamma(n, \alpha, \beta))$, where*

$$\gamma(n, \alpha, \beta) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}. \tag{6.16}$$

The result is sharp for the functions

$$f_1(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \tag{6.17}$$

and

$$f_2(z) = z - \frac{1-\beta}{(2-\beta)(n+1)} z^2. \tag{6.18}$$

Proof. Proceeding as in the proof of Theorem 8, we get

$$\gamma \leq B(k) = \frac{\delta(n, k) - k\left(\frac{1-\alpha}{k-\alpha}\right)\left(\frac{1-\beta}{k-\beta}\right)}{\delta(n, k) - \left(\frac{1-\alpha}{k-\alpha}\right)\left(\frac{1-\beta}{k-\beta}\right)}. \tag{6.19}$$

Since the function $B(k)$ is an increasing function of k $(k \geq 2)$, setting $k = 2$ in (6.19), we obtain

$$\gamma \leq B(2) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}. \tag{6.20}$$

This completes the proof of Theorem 9.

Corollary 5. *Let the functions $f_i(z)$ $(i = 1, 2, 3)$ defined by (5.1) be in the class $R_n^*(\alpha)$, then $f_1 * f_2 * f_3(z) \in R_n^*(\zeta(n, \alpha))$, where*

$$\zeta(n, \alpha) = \frac{(n+1)^2 - 2\left(\frac{1-\alpha}{2-\alpha}\right)^3}{(n+1)^2 - \left(\frac{1-\alpha}{2-\alpha}\right)^3}. \tag{6.21}$$

The result is best possible for the functions

$$f_i(z) = z - \frac{1-\alpha}{(2-\alpha)(n+1)} z^2 \quad (i = 1, 2, 3). \tag{6.22}$$

Proof. From Theorem 8, we have $f_1 * f_2(z) \in R_n^*(\beta)$, where β is given by (6.2). We use now Theorem 9, we get $f_1 * f_2 * f_3(z) \in R_n^*(\zeta(n, \alpha))$, where

$$\zeta(n, \alpha) = \frac{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)}{(n+1) - \left(\frac{1-\alpha}{2-\alpha}\right)\left(\frac{1-\beta}{2-\beta}\right)} = \frac{(n+1)^2 - 2\left(\frac{1-\alpha}{2-\alpha}\right)^3}{(n+1)^2 - \left(\frac{1-\alpha}{2-\alpha}\right)^3}.$$

This completes the proof of Corollary 5.

Theorem 10. *Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (5.1) be in the class $R_n^*(\alpha)$. Then the function*

$$h(z) = z - \sum_{k=2}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{6.23}$$

belongs to the class $R_n^*(\varphi(n, \alpha))$, where

$$\varphi(n, \alpha) = \frac{(n+1) - \left(\frac{2(1-\alpha)}{2-\alpha}\right)^2}{(n+1) - 2\left(\frac{1-\alpha}{2-\alpha}\right)^2}. \tag{6.24}$$

The result is sharp for the functions $f_i(z)$ $(i = 1, 2)$ defined by (6.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 a_{k,1}^2 \leq \left[\sum_{k=2}^{\infty}\frac{(k-\alpha)\delta(n,k)}{1-\alpha}a_{k,1}\right]^2 \leq 1 \tag{6.25}$$

and

$$\sum_{k=2}^{\infty}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 a_{k,2}^2 \leq \left[\sum_{k=2}^{\infty}\frac{(k-\alpha)\delta(n,k)}{1-\alpha}a_{k,2}\right]^2 \leq 1. \tag{6.26}$$

It follows from (6.25) and (6.26) that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 (a_{k,1}^2+a_{k,2}^2) \leq 1. \tag{6.27}$$

Therefore, we need to find the largest $\varphi=\varphi(n,\alpha)$ such that

$$\frac{(k-\varphi)\delta(n,k)}{1-\varphi} \leq \frac{1}{2}\left[\frac{(k-\alpha)\delta(n,k)}{1-\alpha}\right]^2 \quad (k \geq 2), \tag{6.28}$$

that is

$$\varphi \leq \frac{\delta(n,k)-2k\left(\frac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k)-2\left(\frac{1-\alpha}{k-\alpha}\right)^2} \quad (k \geq 2). \tag{6.29}$$

Since

$$D(k)=\frac{\delta(n,k)-2k\left(\frac{1-\alpha}{k-\alpha}\right)^2}{\delta(n,k)-2\left(\frac{1-\alpha}{k-\alpha}\right)^2} \tag{6.30}$$

is an increasing function of k $(k \geq 2)$, we readily have

$$\varphi \leq D(2)=\frac{(n+1)-\left(\frac{2(1-\alpha)}{2-\alpha}\right)^2}{(n+1)-2\left(\frac{1-\alpha}{2-\alpha}\right)^2}, \tag{6.31}$$

and Theorem 10 follows at once.

Theorem 11. *Let $f_1(z) \in R_{n_1}^*(\alpha)$, and $f_2(z) \in R_{n_2}^*(\alpha)$. Then the modified Hadamard product $f_1 * f_2(z) \in R_{n_1}^*(\alpha) \cap R_{n_2}^*(\alpha)$.*

Proof. Since $f_2(z) \in R_{n_2}^*(\alpha)$, we have from (4.3) that

$$a_{k,2} \leq \frac{1-\alpha}{(2-\alpha)(n_2+1)}. \tag{6.32}$$

From Theorem 1, since $f_1(z) \in R_{n_1}^*(\alpha)$, we have

$$\sum_{k=2}^{\infty}\frac{(k-\alpha)\delta(n_1,k)}{1-\alpha}a_{k,1} \leq 1. \tag{6.33}$$

Now, from (6.32) and (6.33), we have

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n_1,k)}{1-\alpha} a_{k,1} a_{k,2} \leq \frac{1-\alpha}{(2-\alpha)(n_2+1)} \sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n_1,k)}{1-\alpha} a_{k,1} \leq$$

$$\leq \frac{1-\alpha}{(2-\alpha)(n_2+1)} \leq 1.$$

Hence $f_1 * f_2(z) \in R^*_{n_1}(\alpha)$. Interchanging n_1 and n_2 by each other in the above, we get $f_1 * f_2(z) \in R^*_{n_2}(\alpha)$. Hence the theorem.

7. **Radii of close-to-convexity, starlikeness and convexity**

Theorem 12. *Let the function $f(z)$ defined by (1.11) be in the class $R^*_n(\alpha)$, then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1(n, \alpha, \rho)$, where*

$$r_1(n,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.1}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. We must show that $|f'(z) - 1| \leq 1 - \rho$ for $|z| < r_1(n, \alpha, \rho)$. We have

$$|f'(z) - 1| \leq \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \leq 1 - \rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k |z|^{k-1} \leq 1. \tag{7.2}$$

Hence, by Theorem 1, (7.2) will be true if

$$\frac{k|z|^{k-1}}{1-\rho} \leq \frac{(k-\alpha)\delta(n,k)}{1-\alpha}$$

or if

$$|z| \leq \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.3}$$

The theorem follows easily from (7.3).

Theorem 13. *Let the function $f(z)$ defined by (1.11) be in the class $R^*_n(\alpha)$, then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2(n, \alpha, \rho)$, where*

$$r_2(n,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho$ for $|z| < r_2(n,\alpha,\rho)$. We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\rho$ if

$$\sum_{k=2}^{\infty}\frac{(k-\rho)a_k|z|^{k-1}}{1-\rho} \le 1. \tag{7.5}$$

Hence, by Theorem 1, (7.5) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{1-\rho} \le \frac{(k-\alpha)\delta(n,k)}{1-\alpha}$$

or if

$$|z| \le \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.6}$$

The theorem follows easily from (7.6).

Corollary 6. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$, then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3(n,\alpha,\rho)$, where*

$$r_3(n,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)(k-\alpha)\delta(n,k)}{k(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{7.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.4).

8. **Integral operators**

Theorem 14. *Let the function $f(z)$ defined by (1.11) be in the class $R_n^*(\alpha)$ and let the function $F(z)$ be defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{8.1}$$

Then

(i) for every c, $c > -1$, $F(z) \in R_n^*(\alpha)$

and

(ii) for every c, $-1 < c \le n$, $F(z) \in R_{n+1}^*(\alpha)$.

Proof. (i) From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{8.2}$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k. \tag{8.3}$$

Therefore,

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)b_k = \sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)\left(\frac{c+1}{c+k}\right) a_k \leq$$

$$\leq \sum_{k=2}^{\infty}(k-\alpha)\delta(n,k)a_k \leq 1-\alpha,$$

since $f(z) \in R_n^*(\alpha)$. Hence, by Theorem 1, $F(z) \in R_n^*(\alpha)$.

(ii) In view of Theorem 1 it is sufficient to show that

$$\sum_{k=2}^{\infty}(k-\alpha)\delta(n+1,k)\left(\frac{c+1}{c+k}\right) a_k \leq 1-\alpha. \tag{8.4}$$

Since

$$\delta(n,k) - \delta(n+1,k)\left(\frac{c+1}{c+k}\right) \geq 0 \text{ if } -1 < c \leq n \ (k = 2, 3, \dots)$$

the result follows from Theorem 1.

Putting $c = 0$ in Theorem 14, we get

Corollary 7. *Let the function $f(z)$ defined by (1.6) be in the class $R_n^*(\alpha)$ and let the function $F(z)$ be defined by*

$$F(z) = \int_0^z \frac{f(t)}{t} dt. \tag{8.5}$$

Then $F(z) \in R_{n+1}^*(\alpha)$.

Theorem 15. *Let the function $F(z) = z - \sum_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$ be in the class $R_n^*(\alpha)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ defined by (8.1) is univalent in $|z| < r^*$, where*

$$r^* = \inf_k \left[\frac{(c+1)(k-\alpha)\delta(n,k)}{k(c+k)(1-\alpha)}\right]^{\frac{1}{k-1}}, \quad (k \geq 2). \tag{8.6}$$

The result is sharp.

Proof. From (8.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} \quad (c > -1) = z - \sum_{k=2}^{\infty}\left(\frac{c+k}{c+1}\right) a_k z^k. \tag{8.7}$$

In order to obtain the required result it suffices to show that

$$|f'(z) - 1| < 1 \text{ in } |z| < r^*.$$

Now

$$|f'(z) - 1| \le \sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$, if

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{8.8}$$

But Theorem 1 confirms that

$$\sum_{k=2}^{\infty} \frac{(k-\alpha)\delta(n,k)}{1-\alpha} a_k \le 1. \tag{8.9}$$

Hence (8.8) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{c+1} < \frac{(k-\alpha)\delta(n,k)}{1-\alpha} \quad (k \ge 2)$$

or if

$$|z| < \left[\frac{(c+1)(k-\alpha)\delta(n,k)}{k(c+k)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \ge 2). \tag{8.10}$$

Therefore $f(z)$ is univalent in $|z| < r^*$. Sharpness follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{(k-\alpha)\delta(n,k)(c+1)} z^k \quad (k \ge 2). \tag{8.11}$$

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