STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

STARLIKE, CONVEX AND ALPHA-CONVEX FUNCTIONS OF HYPERBOLIC COMPLEX AND OF DUAL COMPLEX VARIABLE

SORIN G. GAL

Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

1. Introduction

The study of functions of hyperbolic complex and of dual complex variable was done in [11-12] and continued in the very recent papers [1-6].

In this paper we begin the study of a geometric theory for such of functions, in the general setting of nonanalytic functions.

It is known that for the functions of usual complex variable, the geometric theory is based on the identification of the field of usual complex numbers with the two-dimensional Euclidean plane. But according to the Cayley-Klein scheme, there are nine plane geometries, corresponding to all possible combinations which can be formed for the three kinds of measures of angles and the three kinds of measure of distances (see [13, p. 195-219], [14, p. 214-288]):

1) Elliptic geometry, Euclidean geometry, Hyperbolic geometry, based on the same elliptic (usual) measure of angles but having three different kinds of measures for distances, i.e. elliptic measure, parabolic measures and hyperbolic measure, respectively.

The analytic model for these geometries are the usual complex numbers.

2) Co-Euclidean geometry, Galilean geometry, Co-Minkowskian geometry, based on the same parabolic measure of angles but having the three different kinds of measures for distances as in the case 1, respectively.

The analytic model for these geometries are the dual complex numbers.

3) Cohyperbolic geometry, Minkowskian geometry, doubly hyperbolic geometry, based on the same hyperbolic measure of angles but again having the three different kinds of measures of distances, as above, respectively.

The analytic model for these last three geometries are the hyperbolic complex numbers.

A geometric theory for (analytic) functions of usual complex variable, based on the hyperbolic geometry was done in [7].

In the next sections we will consider a few geometrical aspects for (nonanalytic) functions of hyperbolic complex and of dual complex variables, based on the Minkowskian geometry and on the Galilean geometry, respectively.

Besides the fact that in this way we introduce several plane transformations with new geometrical properties, our method permits an unitary treatment for the geometric theories of functions of usual complex, of hyperbolic complex and of dual complex variables.

Section 2 contains some preliminaries facts.

In the Sections 3 and 4 we introduce and study the classes of starlike, convex and alpha-convex functions of hyperbolic complex and of dual complex variable, respectively.

The methods were suggested by the classical ones in [8-10].

2. **Preliminaries**

First let us recall some known facts about the complex-type numbers (see e.g. [6], [13-14]). It is known that excepting an isomorphism, three kinds of complex numbers are important:

(i) $\mathbb{C}_q$, $q \notin \mathbb{R}$, $q^2 = -1$, called the system of usual complex numbers,

(ii) $\mathbb{C}_q$, $q \notin \mathbb{R}$, $q^2 = 0$, called the system of complex numbers,

(iii) $\mathbb{C}_q$, $q \notin \mathbb{R}$, $q^2 = +1$, called the system of hyperbolic complex numbers, where $\mathbb{C}_q = \{z = x + qy;\ x, y \in \mathbb{R}\}$.

For simplicity, let us denote $q = i$ if $q^2 = -1$, $q = d$ if $q^2 = 0$ and $q = h$ is $q^2 = +1$.

If $q = i$, then $\mathbb{C}_q$ is a field, if $q = d$ then $\mathbb{C}_q$ is a ring with the set of divisors of zero given by $\mathbb{Z}_q = \{z = x + qy;\ x = 0,\ y \in \mathbb{R}\}$ and if $q = h$ then $\mathbb{C}_q$ is a ring with the zero divisors $\mathbb{Z}_q = \{z = x + qy; x, y \in \mathbb{R},\ |x| = |y|\}$. Obviously $\mathbb{Z}_q = \{z \in \mathbb{C}_q;\ \rho_q(z) = 0\}$, where $\rho_q(z)$ is defined below, for all $q \in \{i, d, h\}$.

For $z = x + qy \in \mathbb{C}_q$, let us denote $\overline{z} = x - qy$, (so $z\overline{z} = x^2 - q^2y^2 \in \mathbb{R}$), $\rho_q(z) = \sqrt{|z\overline{z}|}$, for $r > 0$ let us denote $U_r^{(q)} = \{z \in \mathbb{C}_q;\ \rho_q(z) < r\}$, $C_r^{(q)} = \{z \in \mathbb{C}_q;\ \rho_q(z) = r\}$, for all $q \in \{i, d, h\}$.

In the Euclidean geometry, $C_r^{(i)}$ is a circle of radius r and of center $(0,0)$, $C_r^{(d)}$ represents the straigth lines $x = -r$ and $x = +r$, and $C_r^{(h)}$ represents the hyperbolas $x^2 - y^2 = -r^2$, $x^2 - y^2 = r^2$.

The polar coordinates and the exponentials are defined as follows. Let $z = x + qy \in \mathbb{C}_q$. For $q = i$ they are well-known.

For $q = d$ we have $|z|_d = x$, $\varphi = \arg_d z = \dfrac{y}{x}$, $z \notin \mathbb{Z}_d$, and $z = |z|_d(1 + d\varphi) = |z|_d e_d^{d\varphi}$, where $e_d^z = e^x e_d^{dy} = e^x(1 + dy)$.

For $q = h$ we have $e_h^{hy} = \cosh(y) + h\sinh(y)$, $e_h^z = e^x e_h^{hy} = e^x \cosh y + he^x \sinh y$, $|z|_h = (\mathrm{sgn} x)\sqrt{x^2 - y^2}$, $\varphi = \arg_h z = \mathrm{arcth}\dfrac{y}{x}$, $z = |z|_h e_h^{h\varphi}$, for $x^2 - y^2 > 0$, and $|z|_h = (\mathrm{sgn} y)\sqrt{y^2 - x^2}$, $\varphi = \mathrm{arcth}\dfrac{x}{y}$, $z = q|z|_h e_h^{h\varphi}$, for $y^2 - x^2 > 0$. In the first case z is called of first kind (1-kind) and in the other case it is called of second kind (2-kind).

Note that $\mathbb{Z}_q = \{z \in \mathbb{C}_q;\ |z|_q = 0\}$, for all $q \in \{i, d, h\}$.

Let $q \in \{i, d, h\}$ and $\gamma : I \to \mathbb{C}_q$, $\gamma(t) = x(t) + qy(t)$, $t \in I$ (bounded or unbounded interval) be a differentiable path in $\mathbb{C}_q$, such that $\gamma'(t) \notin \mathbb{Z}_q$, $t \in I$. Then $\arg_q[\gamma'(t)]$ represents the "Q"-angle with the positive sense of Ox-axis, of the "Q"-tangent at the path γ in the point $\gamma(t)$, where by convention, everywhere in the paper "Q" means the words Euclidean, Galilean and Minkowskian, for $q = i, d$ and h, respectively.

Let us denote $D_h(f)(z) = z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}$, $\mathcal{D}_h(f)(z) = z\dfrac{\partial f}{\partial z} + \overline{z}\dfrac{\partial f}{\partial \overline{z}}$, $f = U + hV$, $z = x + hy$, where (see [7])

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right] + \frac{h}{2}\left[\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right],$$

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left[\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right] + \frac{h}{2}\left[\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right],$$

i.e.

$$D_h(f)(z) = x\frac{\partial V}{\partial y} + y\frac{\partial V}{\partial x} + h\left[x\frac{\partial U}{\partial y} + y\frac{\partial U}{\partial x}\right],$$

$$\mathcal{D}_h(f)(z) = x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} + h\left[x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right].$$

It is easy to verify the following formulas

$$D_h(\overline{f}) = -\overline{D_h(f)}, \quad \mathcal{D}_h(\overline{f}) = \overline{\mathcal{D}_h(f)}, \quad D_h(\text{Re } f) = h\text{Im } [D_h(f)],$$

$$\mathcal{D}_h[\text{Re } f] = \text{Re } \mathcal{D}_h(f), \quad D_h(\text{Im } h) = h\text{Re } D_h(f), \quad \mathcal{D}_h[\text{Im } f] = \text{Im } \mathcal{D}_h(f),$$

$$\frac{\partial f}{\partial \varphi} = hD_h(f), \quad \frac{\partial f}{\partial |z|_h} = \frac{1}{|z|_h}\mathcal{D}_h(f), \quad D_h(|f|_h) = h|f|_h\text{Im } \frac{D_h(f)}{f},$$

$$\mathcal{D}_h(|f|_h) = |f|_h\text{Re } \frac{\mathcal{D}_h(f)}{f}, \quad D_h(\arg_h f) = h\text{Re } \frac{D_h(f)}{f}, \quad \mathcal{D}_h(\arg_h f) = \text{Im } \frac{\mathcal{D}_h(f)}{f},$$

which immediately imply

$$\frac{\partial |f|_h}{\partial \varphi} = |f|_h\text{Im } \frac{D_h(f)}{f}, \quad \frac{\partial |f|_h}{\partial |z|_h} = \frac{|f|_h}{|z|_h}\text{Re } \frac{\mathcal{D}_h(f)}{f}, \tag{1}$$

$$\frac{\partial \arg_h f}{\partial \varphi} = \text{Re } \frac{D_h(f)}{f}, \quad \frac{\partial \arg_h f}{\partial |z|_h} = \frac{1}{|z|_h}\text{Im } \frac{\mathcal{D}_h(f)}{f}, \tag{2}$$

where in all the above formulas $\varphi = \arg_h z$, $|z|_h \neq 0$, $|f(z)|_h \neq 0$.

Also, if $h \in C^1(\mathbb{R})$, then $D_h(h(z\overline{z})) = 0$ and $\mathcal{D}_h[h(\arg_h z)] = 0$.

Note that these formulas are valid for all the cases when z and $f(z)$ are of first or of second kind. On the other hand, in comparison with the case $q = i$ in [8], among the above formulas only three differ (by sign) from those in [8], namely those which give formulas for $D_h(\text{Im } f)$, $D_h(\arg_h f)$ and $\frac{\partial |f|_h}{\partial \varphi}$.

3. **Starlike functions**

Let $f : U_1^{(q)} \to \mathbb{C}_q$ be of C^1-class on $U_1^{(q)}$, $f = U + qV$, where q is any between i, d and h.

Definition 3.1. We say that f is Symmetrically Uniformly (shortly (SU)) - "Q" starlike function on $U_1^{(q)}$, if f is univalent on $U_1^{(q)} \setminus \mathbb{Z}_q$, $f(z) \in \mathbb{Z}_q$ iff $z \in \mathbb{Z}_q$ and moreover, for any fixed $\rho \in (-1, 1) \setminus \{0\}$, we have

$$\frac{\partial}{\partial \arg_q z}(\arg_q f(z)) > 0, \text{ for all } |z|_q = \rho. \tag{3}$$

The univalency of f is required only on $U_1^{(q)} \setminus \mathbb{Z}_q$ (and not on the whole $U_1^{(q)}$), because the geometric condition in (3) holds only on $U_1^{(q)} \setminus \mathbb{Z}_q$.

Remarks. 1) If $q = i$, then we obtain the classical conditions in [8]: f is (SU)-Euclidean starlike, if f is univalent on the whole $U_1^{(i)}$, $f(0) = 0$ and

$$\operatorname{Re} \frac{D_i f}{f} > 0, \text{ for all } z \in U_1^{(i)} \setminus \{0\}, \tag{4}$$

where $D_i f = z\dfrac{\partial f}{\partial z} - \overline{z}\dfrac{\partial f}{\partial \overline{z}}$ and $\dfrac{\partial f}{\partial z}$, $\dfrac{\partial f}{\partial \overline{z}}$, $z = x + iy$, are given in [8].

¿From [8] it follows that (4) implies the starlikeness of all the sets $f(U_r^{(i)})$, $0 < r < 1$, which suggested us the denomination of "Symmetrically Uniformly" for f.

In fact it is well-known that (see e.g. [10, Theorem 3.1]) if f is analytic and $f'(0) = 0$, then f is (SU)-starlike if and only if f is starlike (in the classical sense).

Since simple calculations show that $D_i(f) = x\dfrac{\partial V}{\partial y} - y\dfrac{\partial V}{\partial x} + i\left(y\dfrac{\partial U}{\partial x} - x\dfrac{\partial U}{\partial y}\right)$ and

$$\operatorname{Re} \frac{D_i(f)}{f} = \frac{1}{U^2 + V^2}\left\{x\left(U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right) + y\left(V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right)\right\},$$

it follows that f generates the injective vectorial transform defined on $U_1^{(i)}$ (in fact on the Euclidean image of $U_1(i)$), $F(x, y) = (U(x, y), V(x, y))$, with $U(0, 0) = V(0, 0) = 0$ and satisfying

$$x\left[U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right] + y\left[V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right] > 0, \ \forall\ x^2 + y^2 \leq 1, \ x \neq 0, \ y \neq 0 \tag{5}$$

(since obviously (4) is equivalent with (5)).

2) Let $q = d$. First, in this case the condition "$f(z) \in \mathbb{Z}_d$ iff $z \in \mathbb{Z}_d$", means that "$U(x, y) = 0$ iff $x = 0$". For $z \in U_1^{(d)} \setminus \mathbb{Z}_d$ we have $z = |z|_d(1 + d\varphi)$, $\varphi = \arg_d z \in \mathbb{R}$, $x = |z|_d = r \neq 0$, $y = r\varphi$ (r fixed in $(-1, 1) \setminus \{0\}$), and (3) becomes

$$\frac{\partial}{\partial \varphi}(\arg_d f) = \frac{\partial}{\partial \varphi}\left(\frac{V}{U}\right) = \frac{UV'_\varphi - VU'_\varphi}{U^2} > 0,$$

where $V'_\varphi = \dfrac{\partial V}{\partial x} \cdot \dfrac{\partial x}{\partial \varphi} + \dfrac{\partial V}{\partial y} \cdot \dfrac{\partial y}{\partial \varphi} = x\dfrac{\partial V}{\partial y}$, $U'_\varphi = x\dfrac{\partial U}{\partial y}$.

As an immediate conclusion it follows that a (SU)-Galilean starlike function f generates the injective vectorial transform on $U_1^{(d)} \setminus \mathbb{Z}_d$, $F(x, y) = (U(x, y), V(x, y))$, with $U(x, y) = 0$ iff $x = 0$ and satisfying

$$x\left(U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right) > 0, \ \forall\ x \in (-1, 1) \setminus \{0\}, \ y \in \mathbb{R}. \tag{6}$$

Note that (6) is equivalent with the inequality

$$x\frac{\partial}{\partial y}\left(\frac{V}{U}\right) > 0, \ \forall\ x \in (-1, 1) \setminus \{0\}, \ y \in \mathbb{R}.$$

3) Let $q = h$. The condition "$f(z) \in \mathbb{Z}_h$ iff $z \in \mathbb{Z}_h$" is equivalent with "$|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$". Let $z \notin \mathbb{Z}_h$, then $\arg_h f(z) = \text{arcth}\dfrac{V}{U}$, for $U^2 - V^2 > 0$ and $\arg_h f(z) = \dfrac{U}{V}$, for $V^2 - U^2 > 0$. Denoting $\arg_h z = \varphi \in \mathbb{R}$, $|z|_h = r \in (-1,1) \setminus \{0\}$ and (3) becomes

$$\left[\text{arcth}\left(\frac{V}{U}\right)\right]'_{\varphi} = \frac{\left(\dfrac{V}{U}\right)'_{\varphi}}{1 - \left(\dfrac{V}{U}\right)^2} = \frac{UV'_{\varphi} - VU'_{\varphi}}{U^2 - V^2} > 0, \text{ if } U^2 - V^2 > 0$$

and

$$\left[\text{arcth}\left(\frac{U}{V}\right)\right]'_{\varphi} = \frac{\left(\dfrac{U}{V}\right)'_{\varphi}}{1 - \left(\dfrac{U}{V}\right)^2} = \frac{UV'_{\varphi} - VU'_{\varphi}}{U^2 - V^2} > 0, \text{ if } V^2 - U^2 > 0.$$

Now, taking into account that for fixed r and independent of the fact that z is of the first or second kind, we have $\dfrac{\partial x}{\partial \varphi} = y$ and $\dfrac{\partial y}{\partial \varphi} = x$, by simple calculations it follows that a (SU)-Minkowskian starlike function f, generates the injective vectorial transform on $U_1^{(h)} \setminus \mathbb{Z}_h$, $F(x,y) = (U(x,y), V(x,y))$ with $|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$, satisfying the differential inequality

$$\frac{1}{U^2 - V^2}\left\{x\left[U\frac{\partial V}{\partial y} - V\frac{\partial U}{\partial y}\right] - y\left[V\frac{\partial U}{\partial x} - U\frac{\partial V}{\partial x}\right]\right\} > 0, \ \forall\, |x^2 - y^2| < 1, \ |x| \neq |y|. \tag{7}$$

On the other hand, taking into account the relations satisfied by $D_h(f)(z)$ in Section 2, we easily obtain that (7) (and therefore (3)) is equivalent with

$$\text{Re}\, \frac{D_h(f)(z)}{f(z)} > 0, \text{ for all } z \in U_1^{(h)} \setminus \mathbb{Z}_h. \tag{8}$$

4) It is immediate that by the conditions in Definition 3.1, f has in addition the following property of univalency: if $z_1 \neq z_2$, $z_1 \in \mathbb{Z}_q$, $z_2 \in U_1^{(q)} \setminus \mathbb{Z}_q$, then $f(z_1) \neq f(z_2)$.

5) The differential inequalities (5), (6), (7), suggest us that each kind of starlikeness in Definition 3.1 is completely independent in respect with the other two, as can be seen in the following simple examples.

Note that in all these examples, U and V are of C^1-class on the whole $\mathbb{R}^2$.

Example 1. Let $U(x,y) = x$, $V(x,y) = x^{100}e^y$. The function $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, is (SU)-Galilean starlike in $U_1^{(d)}$, since it is univalent on $U_1^{(d)} \setminus \mathbb{Z}_d$, $U(x,y) = 0$ iff $x = 0$, and (6) is satisfied. But even if $f(z) = U(x,y) + iV(x,y)$, $z = x + iy$, satisfies $f(0) = 0$, however f cannot be (SU)-Euclidean starlike, because (5) is not satisfied in any $U_r^{(i)}$, $r \in (0,1]$, and f is not univalent on the whole $U_1^{(i)}$.

Also, $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, cannot be (SU)-Minkowskian starlike in $U_1^{(h)}$, firstly because it is not satisfied the condition $|U(x,y)| = |V(x,y)|$ iff $|x| = |y|$, secondly (7) is not satisfied, and thirdly f is not univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$.

Example 2. Let $U(x,y) = x + \dfrac{1}{2}(x^2 - y^2)$, $V(x,y) = y - xy$. By [8], $f(z) = U(x,y) + iV(x,y) = z + \dfrac{1}{2}\overline{z}^2$, $z = x + iy$, is (SU)-Euclidean starlike in $U_1^{(i)}$. But $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, cannot be (SU)-Galilean starlike in $U_1^{(d)}$ (for example, (6) does not hold) and $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, cannot be (SU)-Minkowskian starlike in $U_1^{(d)}$ (for example, (7) does not hold).

Example 3. Let $U(x,y) = xe^{x^2}$, $V(x,y) = ye^{y^2}$. The vectorial function $F(x,y) = (U(x,y), V(x,y))$ is injective on the whole $\mathbb{R}^2$. Let $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$. Then f is (SU)-Galilean starlike on $U_1^{(d)}$, because $U(x,y) = 0$ iff $x = 0$, and (6) becomes

$$x^2 e^{x^2}(1 + 2y^2)e^{y^2} > 0, \text{ for all } x \neq 0, \ y \in \mathbb{R}.$$

Let us denote $g(t) = te^{t^2}$. Since $g'(t) = e^{t^2}(1+2t^2) > 0$, g is strictly increasing on $\mathbb{R}$, and as consequence we obtain $|U(x,y)| = |V(x,y)|$ iff $|x|e^{|x|^2} = |y|e^{|y|^2}$ iff $g(|x|) = g(|y|)$ iff $|x| = |y|$.

The function $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, also is (SU)-Minkowskian starlike on $U_1^{(h)}$, because (7) becomes

$$\frac{e^{x^2} e^{y^2}(x^2 - y^2)}{H(x^2) - H(y^2)} > 0, \text{ for all } x^2 - y^2 \neq 0,$$

taking into account that $H(t) = te^{2t}$ is strictly increasing on $\mathbb{R}_+$.

Now, let us denote $f(z) = U(x,y) + iV(x,y)$, $z = x + iy$. We have $f(0) = 0$ and (5) becomes

$$e^{x^2} e^{y^2}[x^2 + y^2 + 4x^2y^2] > 0, \text{ for all } x \neq 0, \ y \neq 0,$$

which means that f is (SU)-Euclidean starlike too (on $U_1^{(i)}$).

Remark. Let $q = d$ or h. We will say that a region $G \subset \mathbb{C}_q$ is (SU)-"Q" starlike if there exists $f : U_1^{(q)} \to \mathbb{C}_q$ as in Definition 3.1, such that $G = f[U_1^{(q)}]$. Then it would be of interest to give internal geometric characterizations (in Euclidean language) of the (SU)-"Q" starlike regions.

In the following we will obtain some sufficient conditions for (SU)-"Q" starlikeness. Thus, because $U_1^{(d)}$ is an usual convex domain, combining [6, Corollary 3.2] with Definition 3.1 and relation (6), we obtain

Theorem 3.1. *Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x, y) + dV(x, y)$, $z = x + dy$, be of C^1-class. If f satisfies the conditions*

(i) $U(x, y) = 0$ iff $x = 0$,

(ii) $xU\dfrac{\partial V}{\partial y} > 0$ on $U_1^{(d)} \setminus \mathbb{Z}_d$,

(iii) $\dfrac{\partial V}{\partial y} \neq 0$, $\dfrac{\partial U}{\partial x} > 0$, $\dfrac{\partial U}{\partial y} = 0$ on $U_1^{(d)}$ (conditions of univalency),

then f is (SU)-Galilean starlike on $U_1^{(d)}$.

An example of f satisfying Theorem 3.1 is for $U(x, y) = x$, $V(x, y) = (x + 1)^{100}e^y$.

Note that this f is univalent on the whole $U_1^{(d)}$.

Another example is $f(z) = \dfrac{z}{(1 + z)^2}$, $z = x + dy$, which can be written in the form $f = U + dV$, with $U(x, y) = \dfrac{x}{(1 + x)^2}$, $V(x, y) = \dfrac{y(1 - x)}{(1 + x)^3}$.

Now, as in the case $q = i$ in [8], it is of interest to see how the geometric conditions together with the local univalency (imposed by using the Jacobian) could imply the (global) univalency, in the cases $q = d$ and $q = h$ too.

The ideas of proof of Theorem 1 in [8] can be summarized by two properties which must to be checked:

$$f \text{ is univalent on } C_r^{(q)}, \text{ for any fixed } r \in (0, 1), \tag{9}$$

$$f(C_{r_1}^{(q)}) \cap f(C_{r_2}^{(q)}) = \emptyset, \text{ for any } r_1, r_2 \in (0, 1),\ r_1 \neq r_2. \tag{10}$$

But the case $q = i$ is essentially different from the cases $q = d$ and $q = h$, because while for $q = i$, $f(C_r^{(i)})$, $r \in (0, 1)$, are Jordan curves, in the cases $q = d$ and $q = h$ (because of the zero divisors) they are not anymore, which will require additional conditions on f, as can be seen in the following results.

Theorem 3.2. *Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x, y) + dV(x, y)$, $z = x + dy$, be of C^1-class. If f satisfies the conditions:*

(i) $|f(x)|_d = 0$ iff $|z|_d = 0$,

(ii) $J(f)(z) > 0$, *for all* $z \in U_1^{(d)} \setminus \mathbb{Z}_d$, *(here* $J(f)(z)$ *denotes the Jacobian of* f*),*

(iii) $x\dfrac{\partial}{\partial y}\left(\dfrac{V}{U}\right) > 0$, *for all* $x \in (-1,1) \setminus \{0\}$, $y \in \mathbb{R}$,

(iv) Denoting $L_-(x) = \lim\limits_{y\to-\infty} \arg_d f(z)$, $L_+(x) = \lim\limits_{y\to+\infty} \arg_d f(z)$, $\arg_d f(z) = \dfrac{V(x,y)}{U(x,y)}$, $z = x + dy \in U_1^{(d)} \setminus \mathbb{Z}_d$ *(by (iii),* $L_-(x)$, $L_+(x)$ *exist finite or infinite),*

$$I(x) = (L_-(x), L_+(x)) \text{ if } x > 0, \quad I(x) = (L_+(x), L_-(x)) \text{ if } x < 0,$$

and supposing

$$I(\alpha) \cap I(\beta) = \emptyset, \text{ for all } \alpha \in (0,1),\ \beta \in (-1,0), \bigcap_{x\in(0,1)} I(x) \neq \emptyset, \bigcap_{x\in(-1,0)} I(x) \neq \emptyset, \tag{11}$$

then f *is* (SU)*-Galilean starlike on* $U_1^{(d)}$.

Proof. We have to prove that f is univalent on $U_1^{(d)} \setminus \mathbb{Z}_d$. In this sense we will show that for $q = d$, (9) and (10) hold.

For any $r \in (0,1)$ we have $C_r^{(d)} = C_r^{(d^+)} \cup C_r^{(d^-)}$, $C_r^{(d^+)} \cap C_r^{(d^-)} = \emptyset$, where

$$C_r^{(d^+)} = \{z = x + dy;\ x = r\}, \quad C_r^{(d^-)} = \{z = x + dy;\ x = -r\}.$$

Note that $C_r^{(d^+)} \cap \mathbb{Z}_d = \emptyset$, $C_r^{(d^-)} \cap \mathbb{Z}_d = \emptyset$ and that by (i) it follows that $f(C_r^{(d^+)}) \cap \mathbb{Z}_d = \emptyset$, $f(C_r^{(d^-)}) \cap \mathbb{Z}_d = \emptyset$.

In order to prove (9), let $z_1, z_2 \in C_r^{(d)}$, $z_1 \neq z_2$. $r \in (0,1)$ be fixed. If $|z_1|_d = -|z_2|_d$, then by (11) it follows $\arg_d f(z_1) \neq \arg_d f(z_2)$, i.e. $f(z_1) \neq f(z_2)$. So let $|z_1|_d = |z_2|_d$. We have two possibilities:

a) $|z_1|_d = |z_2|_d = r$;

b) $|z_1|_d = |z_2|_d = -r$.

In both cases $\varphi_1 = \arg_d z_1 \neq \arg_d z_2 = \varphi_2$ and by (iii) we get

$$\frac{\partial}{\partial\varphi}[\arg_d f(z)] > 0,\ \varphi = \arg_d z, \text{ i.e. } \arg_d f(z_1) \neq \arg_d f(z_2),$$

which proves (9).

Now, let $r_1, r_2 \in (0,1)$, $r_1 \neq r_2$. We will prove that

$$f(C_{r_1}^{(d^-)}) \cap f(C_{r_2}^{(d^+)}) = \emptyset, \quad f(C_{r_1}^{(d^+)}) \cap f(C_{r_2}^{(d^-)}) = \emptyset \tag{12}$$

and

$$f(C_{r_1}^{(d^+)}) \cap f(C_{r_2}^{(d^+)}) = \emptyset, \quad f(C_{r_1}^{(d^-)}) \cap f(C_{r_2}^{(d^-)}) = \emptyset, \tag{13}$$

which obviously will imply (10).

Indeed, (12) is immediate by (11). Let $\theta \in \bigcap_{x\in(0,1)} I(x)$ be fixed.

For any $\rho \in (0,1)$, by (9) it follows that the system

$$\arg_d f(z) = \theta, \quad |z|_d = \rho \tag{14}$$

yields a unique point $z = \rho e_d^{d\varphi}$, $\varphi = \varphi(\rho)$. Differentiating in respect with ρ, we obtain

$$\left[\frac{\partial}{\partial x}\left(\frac{V}{U}\right)\right](\rho, \rho\varphi(\rho)) + [\rho\varphi(\rho)]' \left[\frac{\partial}{\partial y}\left(\frac{V}{U}\right)\right](\rho, \rho\varphi(\rho)) = 0. \tag{15}$$

On the other hand, for the values of z in (14), denoting $R(\rho) = |f(z)|_d = U(\rho, \rho\varphi(\rho))$, we obtain

$$R'(\rho) = \frac{\partial U}{\partial x}(\rho, \rho\varphi(\rho)) + (\rho\varphi(\rho))' \frac{\partial U}{\partial y}(\rho, \rho\varphi(\rho)). \tag{16}$$

Eliminating $(\rho\varphi(\rho))'$ between (15) and (16) and taking into account (i), (ii) and (iii), we get

$$R'(\rho) = \frac{J(f)(\rho, \rho\varphi(\rho))}{\left[U \dfrac{\partial}{\partial y}\left(\dfrac{V}{U}\right)\right](\rho, \rho\varphi(\rho))} \neq 0, \text{ for all } \rho \in (0,1),$$

i.e. $R'(\rho)$ keeps the same sign on (0,1), which immediately implies that $f(C_{r_1}^{(d^+)}) \cap f(C_{r_2}^{(d^+)}) = \emptyset$.

Now, let $\theta \in \bigcap_{x\in(-1,0)} I(x)$. For any $\rho \in (-1,0)$, reasoning as above, we obtain that $f(C_{r_1}^{(d^-)}) \cap f(C_{r_2}^{(d^-)}) = \emptyset$, which proves (13) and therefore the theorem.

Remarks. 1) From the proof we can see how the geometric condition (iii), together with the condition of local univalency in (ii) imply the global univalency on $U_1^{(d)} \setminus \mathbb{Z}_d$. In comparison with Theorem 1 in [8], because of the zero divisors $\mathbb{Z}_d$ in this case appears the additional condition (11).

2) The function in the previous Example 1 satisfies Theorem 3.2. Another example is $f = U + dV$, with $U(x,y) = x^2$ and $V(x,y) = xe^y$.

Analysing the proof of Theorem 3.2, we see that the condition (11) can be replaced by others. Thus we easily obtain

Corollary 3.1. *Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x,y) + dV(x,y)$, $z = x + dy$, be of C^1-class. If f satisfies the conditions (i), (ii), (iii) in the statement of Theorem*

3.2 and

$$\bigcap_{x\in(0,1)} I(x) \neq \emptyset, \qquad \bigcap_{x\in(-1,0)} I(x) \neq \emptyset, \quad |f(z_1)|_d \neq |f(z_2)|_d,$$

for all $z_1 = x_1+dy_1$, $z_2 = x_2+dy_2 \in U_1^{(d)} \setminus \mathbb{Z}_d$, *with* $x_1x_2 < 0$, *then* f *is* (SU)*-Galilean starlike on* $U_1^{(d)} \setminus \mathbb{Z}_d$.

Remark. The function f in Example 3 and $f(z) = \dfrac{z}{(1+z)^2}$ satisfy Corollary 3.1.

For functions of hyperbolic complex variable we can prove

Theorem 3.3. *Let* $f : U_1^{(h)} \to \mathbb{C}_h$, $f(z) = U(x,y) + hV(x,y)$, $z = x + hy$, *be of* C^1*-class. If* f *satisfies the conditions:*

(i) $|f(z)|_h = 0$ *iff* $|z|_h = 0$,

(ii) $J(f)(z) > 0$, *for all* $z \in U_1^{(h)} \setminus \mathbb{Z}_h$,

(iii) $\operatorname{Re} \dfrac{D_h f(z)}{f(z)} > 0$, *for all* $z \in U_1^{(h)} \setminus \mathbb{Z}_h$,

(iv) $(x^2 - y^2)[U^2(x,y) - V^2(x,y)] > 0$, *on* $U_1^{(h)} \setminus \mathbb{Z}_h$,

(v) if $x_1x_2 < 0$ *then* $U(x_1,y_1)U(x_2,y_2) < 0$ *and if* $y_1y_2 < 0$ *then*

$$V(x_1,y_1)V(x_2,y_2) < 0, \ \text{on } U_1^{(h)} \setminus \mathbb{Z}_h,$$

(vi) Denoting

$$A_1^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to-\infty} \frac{V(sr\cosh\varphi, sr\sinh\varphi)}{U(sr\cosh\varphi, sr\sinh\varphi)}\right],$$

$$B_1^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to+\infty} \frac{V(sr\cosh\varphi, sr\sinh\varphi)}{U(sr\cosh\varphi, sr\sinh\varphi)}\right],$$

$$A_2^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to-\infty} \frac{U(sr\sinh\varphi, sr\cosh\varphi)}{V(sr\sinh\varphi, sr\cosh\varphi)}\right],$$

$$B_2^s(r) = \operatorname{arcth}\left[\lim_{\varphi\to+\infty} \frac{U(sr\sinh\varphi, sr\cosh\varphi)}{V(sr\sinh\varphi, sr\cosh\varphi)}\right],$$

$s \in \{-1,+1\}$, $r \in (0,1)$, *(by (iii), (iv) these numbers exist, finite or infinite and* $A_p^s(r) < B_p^s(r)$, $p \in \{1,2\}$, $s \in \{-1,+1\}$, $r \in (0,1)$*) and supposing that*

$$\bigcap_{r\in(0,1)} (A_p^s(r), B_p^s(r)) \neq \emptyset, \quad p \in \{1,2\}, \ s \in \{-1,+1\},$$

then f *is* (SU)*-Minkowskian starlike on* $U_1^{(h)}$.

Proof. We have to prove that f is univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$, in this sense showing that (9) and (10) hold for $q = h$.

First, it is obvious that for any $r \in (0,1)$ we have

$$C_r^{(h)} = C_r^{(h_1^+)} \cup C_r^{(h_1^-)} \cup C_r^{(h_2^+)} \cup C_r^{(h_2^-)}, \text{ where for } p = 1,2$$

$$C_r^{(h_p^+)} = \{z \in \mathbb{C}_h;\ z \text{ if of } p\text{-kind and } |z|_h = r\},$$

$$C_r^{(h_p^-)} = \{z \in \mathbb{C}_h;\ z \text{ if of } p\text{-kind and } |z|_h = -r\},$$

the four sets being disjoint two by twos.

The univalency of f on each between the above four sets, easily follows from (iii) (since it is equivalent with (3)).

On the other hand, by (iv) we get

$$f(C_r^{(h_1^+)}) \cap f(C_r^{(h_2^+)}) = \emptyset, \quad f(C_r^{(h_1^-)}) \cap f(C_r^{(h_2^-)}) = \emptyset,$$

$$f(C_r^{(h_1^-)}) \cap f(C_r^{(h_2^+)}) = \emptyset, \quad f(C_r^{(h_1^+)}) \cap f(C_r^{(h_2^-)}) = \emptyset,$$

and by (v) we get

$$f(C_r^{(h_1^+)}) \cap f(C_r^{(h_1^-)}) = \emptyset, \quad f(C_r^{(h_2^+)}) \cap f(C_r^{(h_2^-)}) = \emptyset,$$

which immediately proves (9).

Now, let $r_1, r_2 \in (0,1)$, $r_1 \neq r_2$. In order to prove (10), we have to check sixteen relations of the form

$$f(C_{r_1}^{(d_p^s)}) \cap f(C_{r_2}^{(d_l^t)}) = \emptyset, \tag{17}$$

with $p, l \in \{1,2\}$, $s, t \in \{+,-\}$.

For $p \neq l$, (17) follows by (iv). For $s \neq t$, (17) follows by (v). Therefore it remains to prove the following four relations

$$\begin{aligned} f(C_{r_1}^{(h_1^+)}) \cap f(C_{r_2}^{(h_1^+)}) = \emptyset, \quad f(C_{r_1}^{(h_1^-)}) \cap f(C_{r_2}^{(h_1^-)}) = \emptyset, \\ f(C_{r_1}^{(h_2^+)}) \cap f(C_{r_2}^{(h_2^+)}) = \emptyset, \quad f(C_{r_1}^{(h_2^-)}) \cap f(C_{r_2}^{(h_2^-)}) = \emptyset. \end{aligned} \tag{18}$$

In order to obtain the first relation, let $\theta \in (A_1^{+1}, B_1^{+1})$ be fixed.

For any $\rho \in (0,1)$, by (7) we get that the system

$$\arg_h f(z) = \theta, \quad z = x + hy, \quad |z|_h = \rho, \tag{19}$$

yields a unique point $z = \rho e_h^{h\varphi}$, $\varphi = \varphi(\rho)$. For this value of z let us denote $R(\rho) = |f(z)|_h$. We will show that $R(\rho)$, $\rho \in (0,1)$, is strictly monotonous on (0,1), i.e.

$$\frac{d|f|_h}{d|z|_h} = \frac{dR}{d\rho} \text{ keeps the same sign on } (0,1), \tag{20}$$

which will imply the desired conclusion.

In this sense we follow the ideas of proof in [8, Theorem 1].

Differentiating (19) in respect with ρ and using (2), we obtain

$$\frac{1}{\rho}\mathrm{Im}\ \frac{\mathcal{D}_h(f)}{f}+\varphi'(\rho)\mathrm{Re}\ \frac{D_h(f)}{f}=0. \tag{21}$$

Then by (1) we get

$$\frac{dR}{d\rho}=R\left(\frac{1}{\rho}\mathrm{Re}\ \frac{\mathcal{D}_h(f)}{f}+\varphi'(\rho)\mathrm{Im}\ \frac{D_h(f)}{f}\right). \tag{22}$$

Eliminating $\varphi'(\rho)$ between (21) and (22) (since Re $\frac{D_h(f)}{f}\neq 0$), we obtain

$$\frac{dR}{d\rho}\mathrm{Re}\ \frac{D_h(f)}{f}=\frac{R}{\rho}\left[\mathrm{Re}\ \frac{D_h(f)}{f}\mathrm{Re}\ \frac{\mathcal{D}_h(f)}{f}-\mathrm{Im}\ \frac{D_h(f)}{f}\mathrm{Im}\ \frac{\mathcal{D}_h(f)}{f}\right]=$$

$$=\frac{R}{\rho}\mathrm{Re}\ \left[\frac{D_h(f)}{f}\left(\overline{\frac{\mathcal{D}_h(f)}{f}}\right)\right]=\frac{R}{\rho}\cdot\frac{1}{U^2-V^2}\mathrm{Re}\ [D_h(f)\cdot\overline{\mathcal{D}_h(f)}].$$

Since by direct calculation Re $[D_h(f)\cdot\overline{\mathcal{D}_h(f)}]=(x^2-y^2)J(f)$, we get the formula

$$\frac{dR}{d\rho}\mathrm{Re}\ \frac{D_h(f)}{f}=\frac{R}{\rho}\cdot\frac{x^2-y^2}{U^2-V^2}J(f),$$

which can be written in the form

$$\frac{d|f(z)|_h}{d|z|_h}\mathrm{Re}\ \frac{D_h(f)}{f}=\frac{|f(z)|_h}{|z|_h}\cdot\frac{x^2-y^2}{U^2-V^2}J(f). \tag{23}$$

As conclusion, the sign of $\dfrac{d|f(x)|_h}{d|z|_h}$ is the same with the sign of $\dfrac{|f(z)|_h}{|z|_h}$. But because $U_1^{(h_1^+)}=\{z=x+hy\in U_1^{(h)};\ x^2-y^2>0,\ x>0\}$ is obviously a connected set (in $\mathbb{R}^2$), by the hypothesis it follows that the continuous function $F: U_1^{(h_1^+)}\to\mathbb{R}$, $F(z)=\dfrac{|f(z)|_h}{\rho}=\dfrac{|f(z)|_h}{|z|_h}$ keeps the same sign on $U_1^{(h_1^+)}$, which proves the first relation in (18).

Taking now $\theta\in(A_1^{-1},B_1^{-1})$ and again considering (18) but for $\rho\in(-1,0)$, by similar reasonings we obtain (23), which will imply that $f(C_{r_1}^{(h_1^-)})\cap f(C_{r_2}^{(h_1^-)})=\emptyset$, $r_1\neq r_2$.

Analogously we can prove the last two relations in (18), which completes the proof.

Remarks. 1) The previous Example 3 satisfies Theorem 3.3, while $f(z)=z^2\overline{z}$ do not satisfies it, but still is starlike.

2) By the relations $\cosh\varphi = \sqrt{1+\sinh^2\varphi}$ and denoting $\sinh\varphi = t$, it is easy to see that the conditions in Theorem 3.3,(vi), can be written as

$$A_1^s(r) = \operatorname{arcth}\left[\lim_{t\to-\infty}\frac{V(sr\sqrt{1+t^2}, srt)}{U(sr\sqrt{1+t^2}, srt)}\right],$$

$$B_1^s(r) = \operatorname{arcth}\left[\lim_{t\to+\infty}\frac{V(sr\sqrt{1+t^2}, srt)}{U(sr\sqrt{1+t^2}, srt)}\right],$$

and similarly for $A_2^s(r)$, $B_2^s(r)$, $s \in \{-1,+1\}$, $r \in (0,1)$.

3) Condition (iv) in Theorem 3.3 assures that the kind of $z \in U_1^{(h)}$ is not changed by f. On the other hand, it is obvious that (iv), (v), (vi) can be replaced by other conditions.

4. **Convex and alpha-convex functions**

Let q be any between i, d, h, $f : U_1^{(q)} \to \mathbb{C}_q$, $f(z) = U(x,y) + qV(x,y)$, $z = x + qy$, f of C^2-class on $U_1^{(q)}$. For any fixed $r \in (0,1)$, let us consider the differentiable path in $\mathbb{C}_q$, $\gamma_q(\varphi) = f(C_r^{(q)})$, where $\varphi = \arg_q z$ is variable and $|z|_q$ is constant ($|z|_q = r$ if $q = i$, $|z|_q = \pm r$ if $q = d, h$).

Then

$$\gamma_q'(\varphi) = \frac{\partial U}{\partial x}\cdot\frac{\partial x}{\partial\varphi} + \frac{\partial U}{\partial y}\cdot\frac{\partial y}{\partial\varphi} + q\left[\frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial\varphi} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial\varphi}\right], \tag{24}$$

and $\arg_q[\gamma_q'(\varphi)]$ represents the "Q"-angle (with the positive sense of Ox-axis) of the "Q"-tangent at the path $f(C_r^{(q)})$ in $\gamma_q(\varphi)$.

Definition 4.1. We say that f is (SU)-"Q" convex on $U_q^{(q)}$ if f is univalent on $U_1^{(q)} \setminus \mathbb{Z}_q$, $\gamma_q'(\varphi) \in \mathbb{Z}_q$ iff $z \in \mathbb{Z}_q$ and moreover, for any fixed r with $A_r = \{z \in \mathbb{C}_q;\ |z|_q = r\} \cap (U_1^{(q)} \setminus \mathbb{Z}_q) \neq \emptyset$, we have

$$\frac{\partial}{\partial\varphi}(\arg_q \gamma_q'(\varphi)) > 0, \text{ for all } z \in A_r. \tag{25}$$

Remarks. 1) Let $q = i$. Then by (24) and by $x = r\cos\varphi$, $y = r\sin\varphi$, $\varphi \in (0, 2\pi]$, we get that (25) is equivalent with the inequality $\dfrac{\partial}{\partial\varphi}[\arg_i D_i(f)] > 0$, and we obtain the equivalent inequality in [8]

$$\operatorname{Re}\frac{D_i^2(f)(z)}{D_i(f)(z)} > 0, \quad z \in U_1^{(i)} \setminus \{0\}.$$

2) Let $q = d$. In this case $z = x(1 + d\varphi)$, where $x = \pm r$, $y = x\varphi$, $\varphi \in \mathbb{R}$,

$$\gamma_d'(\varphi) = x\frac{\partial U}{\partial y} + q\left[x\frac{\partial V}{\partial y}\right], \quad \arg_d(\gamma_d'(\varphi)) = \frac{\partial V}{\partial y}/\frac{\partial U}{\partial y},$$

for $x \neq 0$, and simple calculations show that a (SU)-Galilean convex function f, generates the injective vectorial transform on $U_1^{(d)} \setminus \mathbb{Z}_d$, $F(x, y) = (U(x, y), V(x, y))$, with $\frac{\partial U}{\partial y}(x, y) = 0$ iff $x = 0$ and satisfying

$$x\left(\frac{\partial U}{\partial y} \cdot \frac{\partial^2 V}{\partial y^2} - \frac{\partial V}{\partial y} \cdot \frac{\partial^2 U}{\partial y^2}\right) > 0, \ \forall\, x \in (-1, 1) \setminus \{0\}, \ y \in \mathbb{R}. \tag{26}$$

Obviously that (26) is equivalent with

$$x\frac{\partial}{\partial y}[(\partial V/\partial y)/(\partial U/\partial y)] > 0, \quad x \in (-1, 1) \setminus \{0\}, \ y \in \mathbb{R}.$$

A simple example of (SU)-Galilean convex function is $f = U + dV$, with $U(x, y) = xe^y$, $V(x, y) = -y$. Note that f is univalent on the whole $U_1^{(d)}$.

3) Let $q = h$. In this case, we obtain: $z = |z|_h(\cosh\varphi + h\sinh\varphi)$ if z is of first kind, $z = |z|_h(\sinh\varphi + h\cosh\varphi)$ if z is of second kind, $\varphi \in \mathbb{R}$, $|z|_h = \pm r$ (constant), and in both cases $\frac{\partial x}{\partial\varphi} = y$, $\frac{\partial y}{\partial\varphi} = x$.

Then by (24) we obtain

$$\gamma_h'(\varphi) = x\frac{\partial U}{\partial y} + y\frac{\partial U}{x} + h\left(x\frac{\partial V}{\partial y} + y\frac{\partial V}{\partial x}\right) = qD_h(f)(z),$$

which immediately implies $\arg_h[\gamma_h'(\varphi)] = \arg_h[D_h(f)(z)]$.

Reasoning exactly as in the case of starlikeness, we can say that f is (SU)-Minkowskian convex on $U_1^{(h)}$, if f is univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$, $D_h(f)(z) \in \mathbb{Z}_h$ iff $z \in \mathbb{Z}_h$ and

$$\operatorname{Re}\frac{D_h^2(f)(z)}{D_h(f)(z)} > 0, \text{ for all } z \in U_1^{(h)} \setminus \mathbb{Z}_h. \tag{27}$$

A simple example of (SU)-Minkowskian convex function is $f(z) = z^2\overline{z}$, $z = x + hy$.

4) Let $q = d$ or h. We will say that a region $G \subset \mathbb{C}_q$ is (SU)-"Q" convex, if there exists $f : U_1^{(q)} \to \mathbb{C}_q$, (SU)-"Q" convex function on $U_1^{(q)}$ such that $G = f(U_1^{(q)})$. An interesting question would be to find internal geometric characterization (in Euclidean language) of the (SU)-"Q" convex regions.

By using the ideas in [9], at end we can introduce the concept of alpha-convex functions.

The Remarks 2 after the Definitions 3.1 and 4.1, suggest

Definition 4.2. Let $f : U_1^{(d)} \to \mathbb{C}_d$, $f(z) = U(x, y) + dV(x, y)$, $z = x + dy$, be of C^2-class on $U_1^{(d)}$ and α a real number. The function f is called (SU)-Galilean α-convex if f is univalent on $U_1^{(d)} \setminus \mathbb{Z}_d$, $U(x, y) = 0$ iff $x = 0$, $\dfrac{\partial U}{\partial y}(x, y) = 0$ iff $x = 0$, and for all $x \in (-1, 1) \setminus \{0\}$, $y \in \mathbb{R}$, we have

$$(1-\alpha)\frac{\partial[D(U,V)]}{\partial y} + \alpha\frac{\partial\left[D\left(\dfrac{\partial U}{\partial y}, \dfrac{\partial V}{\partial y}\right)\right]}{\partial y} > 0,$$

where $D(U, V) = x\left(\dfrac{V}{U}\right)$.

Note that $f(z) = U(x, y) + dV(x, y)$, $z = x + dy$, with $U(x, y) = xe^y$, $V(x, y) = e^{2y}$ is (SU)-Galilean α-convex, for any $\alpha > -1$.

By the relations (8) and (27) we can introduce

Definition 4.3. Let $f : U_1^{(h)} \to \mathbb{C}_h$, $f(z) = U(x, y) + hV(x, y)$, $z = x + hy$, be of C^2-class on $U_1^{(h)}$ and α a real number. The function f is called (SU)-Minkowskian α-convex if f is univalent on $U_1^{(h)} \setminus \mathbb{Z}_h$, $|U(x, y)| = |V(x, y)|$ iff $|x| = |y|$, $\left|x\dfrac{\partial V}{\partial y} + y\dfrac{\partial V}{\partial x}\right| = \left|x\dfrac{\partial U}{\partial y} + y\dfrac{\partial U}{\partial x}\right|$ iff $|x| = |y|$ and on $U_1^{(h)} \setminus \mathbb{Z}_h$ we have

$$\text{Re}\ \left[(1-\alpha)\frac{D_h(f)(z)}{f(z)} + \alpha\frac{D_h^2(f)(z)}{D_h(f)(z)}\right] > 0.$$

Note that $f(z) = z^2\overline{z}$, $z = x + hy$, is (SU)-Minkowskian α-convex, for any $\alpha \in \mathbb{R}$.

Remark. A deeper study of the function classes introduced in this paper together with a corresponding theory for spirallike functions will be done in another paper.

Also, the method in this paper can be extended to functions of hypercomplex variables, as for example of quaternionic variable, or even in abstract Clifford algebras, and will be done elsewhere.

References

[1] P. Fjelstad and S.G. Gal, *n-dimensional dual complex numbers*, Adv. Appl. Clifford Algebras, 8, 2(1998) (in press).

[2] P. Fjelstad and S.G. Gal, *n-dimensional hyperbolic complex numbers*, Adv. Appl. Clifford Algebras, 8, 1(1998), 47-68.

[3] P. Fjelstad and S.G. Gal, *Note on the monogeneity of functions of complex-type variable*, submitted.

[4] S.G. Gal, *On the sufficient conditions of monogeneity for functions of complex-type variable*, Rev. Un. Mat. Argentina, 40(1996), 33-42.

[5] S.G. Gal, *Approximation and interpolation of functions of hyperbolic complex variable*, Rev. Un. Mat. Argentina, 40(3-4)(1997), 25-35.

[6] S.G. Gal, *Univalent functions of hyperbolic complex and of dual complex variable*, Mathematica (Cluj), submitted.

[7] W. Ma and D. Minda, *Hyperbolically convex functions*, Ann. Polon. Math., 60(1994), 81-100.

[8] P.T. Mocanu, *Starlikeness and convexity for non-analytic functions in the unit disc*, Mathematica (Cluj), 22(45), 1(1980), 77-83.

[9] P.T. Mocanu, *Alpha-convex nonanalytic functions*, Mathematica (Cluj), 29(52), 1(1987), 49-55.

[10] P.T. Mocanu, *Geometrical aspects in the theory of functions of complex variable* (in Romanian), in Complex Analysis, Classical and Modern Aspects (C. Andreian-Cazacu Ed.), Ed. St. Encicl., Bucharest, 1988, 17-34.

[11] J.C. Vignaux, *The theory of functions of one and of several dual complex variables* (in Spanish), Univ. Nac. La Plata, Publ. Fac. Ci. Fisicomat. Contrib., 107(1936), 221-282; 109(1937), 389-406.

[12] J.C. Vignaux and A. Durañona Vedia, *On the theory of functions of a hyperbolic complex variable* (in Spanish), Univ. Nac. La Plata, Publ. Fac. Ci. Fisicomat. Contrib., 104(1935), 139-183.

[13] I.M. Yaglom, *Complex Numbers in Geometry*, Academic Press, New York, 1968.

[14] I.M. Yaglom, *A Simple non-Euclidean Geometry and its Physical Basis*, Springer-Verlag, New York, 1979.

Department of Mathematics, University of Oradea, Str. Armatei Române 5, 3700 Oradea, Romania e-mail: galso@hs.uoradea.ro