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ON 6-DIMENSIONAL HERMITIAN SUBMANIFOLDS OF CAYLEY ALGEBRA

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Abstract. Results have been obtained concerning one of the most important characteristics of a Hermitian manifold which is a Ricci curvature.

One of the most beautiful and substantial examples of Hermitian manifolds are the 6-dimensional oriented submanifolds of Cayley octave algebra. In the present work a number of results on the characteristics of such manifolds are shown. Let's remember that Hermitian is named the manifold M^{2n}, that has an almost complex structure J and Riemannian metric $g = \langle \cdot, \cdot \rangle$ when meeting the conditions:

1) $\langle JX, JY \rangle = \langle X, Y \rangle, \; X, Y \in \aleph(M)$;

2) $[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0$.

1.

It is well-known that Ricci tensor ric of a Riemannian manifold is named the tensor whose components are connected with the components of the tensor of Riemannian curvature (Riemann-Christoffel tensor) as follows [4]:

$$ric_{ij} = R^k_{ijk}.$$

This tensor is symmetric; the value of the corresponding quadratic form on vector X, $X \in \aleph(M)$ is called Ricci curvature and is denoted $S(X)$. Thus,

$$S(X) = ric_{ij} X^i X^j, \quad \|X\| = 1.$$

Let's use the values of the spectrum of the Riemann-Christoffel tensor of 6-dimensional Hermitian submanifolds of octave algebra [1].

$$R_{abcd} = R_{\hat{a}bcd} = R_{\widehat{\hat{a}b}cd} = 0;$$

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$$R_{\hat{a}b\hat{c}d} = -\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{bd}, \tag{1}$$

where T^{φ}_{ij} are the components of configuration tensor (or, in other words, the tensor of Euler curvature). Here $a, b, c, d = 1, 2, 3$; $\hat{a} = a + 3$; $\varphi = 7, 8,\ i, j = 1, 2, 3, 4, 5, 6$.

Let's calculate the Ricci tensor spectrum for the 6-dimensional Hermitian submanifolds of Cayley algebra. Taking into account (1), we get:

$$ric_{ab} = R^{c}_{abc} + R^{\hat{c}}_{ab\hat{c}} = R_{\hat{c}abc} + R_{cab\hat{c}} = 0;$$

$$ric_{\hat{a}b} = R^{c}_{\hat{a}bc} + R^{\hat{c}}_{\hat{a}b\hat{c}} = R_{\hat{c}\hat{a}bc} + R_{c\hat{a}b\hat{c}} =$$

$$= R_{c\hat{a}b\hat{c}} = R_{\hat{a}c\hat{c}b} = -\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{cb};$$

$$ric_{a\hat{b}} = R^{c}_{a\hat{b}c} + R^{\hat{c}}_{a\hat{b}\hat{c}} = R_{\hat{c}a\hat{b}c} + R_{ca\hat{b}\hat{c}} =$$

$$= R_{\hat{c}a\hat{b}c} = -\sum_{\varphi} T^{\varphi}_{\hat{c}\hat{b}} T^{\varphi}_{ac};$$

$$ric_{\hat{a}\hat{b}} = R^{c}_{\hat{a}\hat{b}c} + R^{\hat{c}}_{\hat{a}\hat{b}\hat{c}} = R_{\hat{c}\hat{a}\hat{b}c} + R_{c\hat{a}\hat{b}\hat{c}} = 0.$$

In view of the reality of the Ricci tensor,

$$ric_{ab} = \overline{ric_{\hat{a}\hat{b}}}; \quad ric_{\hat{a}b} = \overline{ric_{a\hat{b}}}.$$

Therefore, the Ricci tensor spectrum is calculated as follows:

$$ric_{ab} = 0; \quad ric_{\hat{a}b} = -\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{bc}. \tag{2}$$

Then, the Ricci curvature of Hermitian 6-dimensional submanifolds of octave algebra is calculated as follows:

$$S(X) = -2\sum_{\varphi} T^{\varphi}_{\hat{a}\hat{c}} T^{\varphi}_{bc} X^{b} X_{a} = -2\sum_{\varphi} (T^{\varphi}_{\hat{a}\hat{c}} X_{a})(T^{\varphi}_{bc} X^{b}) =$$

$$= -2\sum_{\varphi} (T^{\varphi}_{ab} X^{b})(\overline{T^{\varphi}_{ab} X^{b}}) = -2\sum_{\varphi, a, b} |T^{\varphi}_{ab} X^{b}|^{2}.$$

Thus,

$$S(X) = -2\sum_{\varphi, a, b} |T^{\varphi}_{ab} X^{b}|^{2},$$

and so, we conclude that is correct

Theorem 1. *The 6-dimensional Hermitian submanifold of Cayley algebra has a nonpositive Ricci curvature, moreover the above mentioned curvature vanishes in geodesic points and only in them.*

Consequence. *The 6-dimensional Hermitian submanifold of Cayley algebra is Ricci flat manifold then and only then, when it is a domain on the Kählerian plane.*

2.

Let's calculate the scalar curvature of the 6-dimensional Hermitian submanifolds of Cayley algebra. Taking into account (2) we get

$$K = ric_i^i = -2 \sum_{\varphi, a, b} |T_{ab}^{\varphi}|^2 \leq 0.$$

Evidently, the scalar curvature of the 6-dimensional Hermitian submanifolds of octave algebra is also nonpositive and becomes zero exclusively in geodesic points. In this sense, the scalar curvature "repeats" both the Ricci curvature and the bisectional holomorphic curvature [2] of such manifolds.

If the considered manifold is a manifold of constant scalar curvature ($K = const$), then we get, that

$$\sum_{\varphi, a, b} |T_{ab}^{\varphi}|^2 = const$$

and therefore is correct.

Theorem 2. *The 6-dimensional Hermitian submanifold of Cayley algebra is a manifold of the constant scalar curvature in the case and only when the configuration tensor has a constant length.*

Let's note that both the theorems sum up the well-known results obtained by V. Kirichenko [3] on the 6-dimensional Kahlerian submanifolds of octave algebra.

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