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INEQUALITIES FOR GENERALIZED CONVEX FUNCTIONS WITH APPLICATIONS II

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In the first part [23] of this series on Inequalities for Generalized Convexity we have studied the most important results and ideas of the author (and coauthors) related to the Jensen inequality. In this part we shall study Hadamard's (or Jensen-Hadamard's, or Hermite-Hadamard's) integral inequality for convex or generalized convex functions. This inequality was applied for the first time by Hadamard in the study of the Riemann zeta function [4]. Many new applications in geometry, special functions, number theory, theory of means, etc. have been published by the author (for References, see [9-25] and Part I). We plan to publish in Part IV of these series some of these applications (Part III will be devoted to Jessen's inequality). As we have stated in the first part [23], in many cases only the new results will be presented with a proof; the other results will be stated only, with connections and/or applications to known theorems. In the course of this survey many new results, new connections, hints, or applications will be pointed out.

2. Hadamard's inequality

Let $f : [a, b] \to \mathbb{R}$ be a convex function (in the classical sense). Then Hadamard's inequality (or "inequalities") states that

$$(b-a)f\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx \leq (b-a)\left[\frac{f(a)+f(b)}{2}\right]. \tag{1}$$

This is in fact Corollary 1.1 of Theorem 1.1 from [23]. In the literature (which is quite extensive) there exist papers where the left-side of (1) is called as "Jensen's inequality", while the right-side is due to Hadamard (or vice-versa). In the last time many papers call (1) as the Hermite-Hadamard inequality, since it seems that Hermite was the first discoverer of these relations ([6]). In that period, Jensen also has an important role in the theory of convexity and inequalities of type (1) ([16]).

A. The first extension of the left side of (1) for generalized convex functions has been discovered in 1982 by the author.

Theorem 2.1. ([9]) *Let $f \in C^{2k}[a,b]$ ($k \geq 1$, integer) be a $2k$-convex function on (a,b). Then*

$$\sum_{j=0}^{k-1} \frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!} f^{(2j)}\left(\frac{a+b}{2}\right) \leq \int_a^b f(x)dx. \tag{2}$$

This result became widely known after its publication in an international journal [10].

For a particular case, namely $k = 2$ one gets:

Corollary 2.1. *Let $f : [a,b] \to \mathbb{R}$, $f \in C^4[a,b]$ and $f^{(4)}(t) \underset{(>)}{\geq} 0$ on (a,b). Then*

$$\int_a^b f(x)dx \underset{(>)}{\geq} (b-a)\left[f\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{24} f''\left(\frac{a+b}{2}\right)\right]. \tag{3}$$

Remark 2.1. To show the power of this inequality, let us consider, as an immediate application, $a > 0$, $b = a+1$ and let $f_1(x) = \dfrac{1}{x}$, $f_2(x) = -\ln x$ $(x > 0)$ which fulfill the above conditions. After certain elementary computations one can deduce the double-inequality

$$\frac{2a+2}{2a+1} e^{1/6(2a+1)^2} < \frac{e}{\left(1+\dfrac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \cdot e^{-1/3(2a+1)^2} \tag{4}$$

for all real numbers $a > 0$. Clearly, this implies the weaker relations

$$\frac{2a+2}{2a+1} < \frac{e}{\left(1+\dfrac{1}{a}\right)^a} < \sqrt{1+\frac{1}{a}} \tag{5}$$

which in turn are quite strong to imply, or improve certain known results. For example, the much studied inequality by Pólya and Szegö [8], namely

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1} \quad (n \geq 1, \text{ integer}) \tag{6}$$

follows immediately, even in improved form from (5). All inequalities of [2] are particular cases, or implications of relations (5). For applications to Stirling's theorem and other inequalities for the number e we quote the recent papers [21], [24], [25]. We note that when f is strictly $2k$-convex, we have strict inequality in (2) (the same in the particular case (3)).

In 1989 H. Alzer [1] has extended the right side of (1):

Theorem 2.2. *Let f be as in Theorem 2.1. Then*

$$\int_a^b f(x)dx \le \frac{1}{2}\sum_{i=1}^{2k-1}\frac{(b-a)^i}{i!}[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)] \tag{7}$$

When f is strictly $2k$-convex, then (7) holds true with strict inequality.

Remark 2.2. By using (7), the following rational approximation of the exponential function can be deduced ([1]):

For all $x>0$ and all integers $n\ge 0$ we have

$$\frac{1+\frac{1}{2}\sum_{i=0}^{2n}\frac{(-x)^{i+1}}{(i+1)!}}{1+\frac{1}{2}\sum_{i=0}^{2n}\frac{x^{i+1}}{(i+1)!}} < e^{-x} < \frac{1+\frac{1}{2}\sum_{i=0}^{2n+1}\frac{(-x)^{i+1}}{(i+1)!}}{1+\frac{1}{2}\sum_{i=0}^{2n+1}\frac{x^{i+1}}{(i+1)!}} \tag{8}$$

Inequalities of this type have applications in irrationality proofs (see [11]).

In 1991 the author obtained common generalizations of Theorem 2.1 and 2.2.

Theorem 2.3. ([17]) *Let f be as in Theorem 2.1. Let $t\in[a,b]$ arbitrary chosen. Then*

$$\int_a^b f(x)dx \ge \sum_{i=1}^{2k}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right]\cdot(-1)^{i-1}f^{(i-1)}(t)+$$

$$+\frac{1}{(2k)!}\left\{\frac{(b-a)^{2k}}{2^{2k-1}}[f^{(2k-1)}(t)-f^{(2k-1)}(a)]+S_{k,a,b}(t)\right\}, \tag{9}$$

respectively

$$\int_a^b f(x)dx \le \sum_{i=1}^{2k}\left[\frac{(t-a)^i-(t-b)^i}{i!}\right]\cdot(-1)^{i-1}f^{(i-1)}(t)+$$

$$+\frac{1}{(2k)!}\{(b-a)^{2k}[f^{(2k-1)}(t)-f^{(2k-1)}(a)]+S_{k,a,b}(t)\}, \tag{10}$$

where

$$S_{k,a,b}(t)=\int_a^b(b-x)^{2k}f^{(2k)}(x)dx-2\int_a^b(b-x)^{2k}f^{(2k)}(x)dx.$$

When f is strictly $2k$-convex, then all inequalities in (9) and (10) are strict.

Remark 2.3. Clearly, this result has a lot of particular cases. For example, by putting $t = a$ and $t = b$ resp. in (10), after addition we get Alzer's inequality (7). By doing the same thing in (9), we get

Theorem 2.4. *With the same conditions,*

$$\frac{1}{2}\sum_{i=1}^{2k}\frac{(b-a)^i}{i!}[f^{(i-1)}(a)+(-1)^{i-1}f^{(i-1)}(b)]+$$
$$+\frac{(b-a)^{2k}}{2^{2k-2}(2k)!}[f^{(2k-1)}(b)-f^{(2k-1)}(a)] \le \int_a^b f(x)dx. \tag{11}$$

By applying (9) and (10) for $t = \frac{a+b}{2}$, and remarking that

$$(x-a)^{2k} \le \left(\frac{b-a}{2}\right)^{2k} \text{ for } x \in \left[a, \frac{a+b}{2}\right],$$

while

$$(b-x)^{2k} \le \left(\frac{b-a}{2}\right)^{2k} \text{ for } x \in \left[\frac{a+b}{2}, b\right],$$

we get firstly our result (2) as well as the following:

Theorem 2.5. *With the same conditions,*

$$\int_a^b f(x)dx \le \sum_{j=0}^{k-1}\frac{(b-a)^{2j+1}}{2^{2j}(2j+1)!}f^{(2j)}\left(\frac{a+b}{2}\right)+$$
$$+\frac{1}{(2k)!2^{2k}}(b-a)^{2k}[f^{(2k-1)}(b)-f^{(2k-1)}(a)]. \tag{12}$$

In what follows, let us use the following notations:

$$A_k \equiv A_k(a,b,f) = f^{(2k-1)}(b) - f^{(2k-1)}(a);$$

$$B_k = B_k(a,b,f) = f^{(k-1)}(a) + (-1)f^{(k-1)}(b).$$

The following two auxiliary results will be necessary:

Lemma 2.1. ("Green-Lagrange identity") *For $f, g \in C^n[a,b]$ one has the identity*

$$\int_a^b g^{(n)}(x)f(x)dx = [g^{(n-1)}(x)f(x) - \cdots + (-1)^{n-1}g(x)f^{(n-1)}(x)]\Big|_a^b+$$
$$+(-1)^n\int_a^b g(x)f^{(n)}(x)dx. \tag{13}$$

Lemma 2.2. (Chebisheff's integral inequality) *Let $u, v : [a, b] \to \mathbb{R}$ be two synchrone functions (i.e. functions having the same type of monotonicity). Then*

$$\frac{1}{b-a}\int_a^b u(x)v(x)dx \geq \frac{1}{b-a}\int_a^b u(x)dx \cdot \frac{1}{b-a}\int_a^b v(x)dx \tag{14}$$

When u and v are asynchrone functions (having different type of monotonicity), then the inequality sign in (14) is reversed. It is known that equality holds in (14) only when one of u and v is constant on $[a, b]$, eventually excepting a numerable subset of $[a, b]$ (see [5]).

We now are able to state the following result:

Theorem 2.6. *Let $f^{(2k)}$ ($k \geq 1$, integer) be a continuous, decreasing function on $[a, b]$. Then*

$$\int_a^b f(x)dx \leq \sum_{j=1}^{2k} \frac{1}{2^j \cdot j!}(b-a)^j B_j + \frac{1}{2^{2k}} \cdot \frac{(b-a)^{2k}}{(2k+1)!}A_k. \tag{15}$$

If $f^{(2k)}$ is monotone increasing, then the sign of inequality in (15) reverses.

Proof. Let $g(x) = \left(x - \frac{a+b}{2}\right)^n$ in (13). By remarking that

$$g^{(k)}(x) = n(n-1)\dots(n-k+1)\left(x - \frac{a+b}{2}\right)^{n-k},$$

after certain elementary computations one can deduce the following identity

$$\int_a^b f(x)dx = \sum_{j=1}^{n} \frac{1}{2^j \cdot j!}B_j + \frac{(-1)^n}{n!}\int_a^b \left(x - \frac{a+b}{2}\right)^n f^{(n)}(x)dx. \tag{16}$$

Let now $n := 2k$ in (16) and put $u(x) := f^{(2k)}(x)$, $v(x) := \left(x - \frac{a+b}{2}\right)^{2k}$ in (14). Since u and v are monotone increasing functions, we have

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2k} f^{(2k)}(x)dx \leq \frac{1}{2^{2k}(2k+1)!}(b-a)^{2k}A_k,$$

and the result follows.

Theorem 2.7. *Let $f^{(2k-1)}$ be increasing and continuous on $[a, b]$. Then*

$$\int_a^b f(x)dx \leq \sum_{j=1}^{2k-1} \frac{1}{2^j \cdot j!}(b-a)^j B_j \tag{17}$$

When $f^{(2k-1)}$ is decreasing, (17) holds true with reversed inequality.

Proof. Apply (16) with $n := 2k + 1$ and put

$$u(x) := f^{(2k-1)}(x), \quad v(x) := \left(x - \frac{a+b}{2}\right)^{2k-1}.$$

Remarking that

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2k-1} dx = 0$$

we obtain from Lemma 2.2 that

$$\int_a^b \left(x - \frac{a+b}{2}\right)^{2k-1} f^{(2k-1)}(x)dx \le 0$$

and (17) follows.

B. Hadamard's inequality has the following geometrical interpretation: the area below the graph of f on $[a, b]$ lies between the areas of two trapeziums, namely the one formed by the points of coordinates $(a, f(a))$; $(b, f(b))$ with the Ox axis, the second one formed by the tangent to the graph of f at the point $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$ with the Ox axis. By rotating these trapeziums round about the Ox axis, we get three volumes,

$$V = \pi \int_a^b f^2(x)dx,$$

$$V_1 = \frac{\pi(b-a)}{3}[f^2(a) + f(a)f(b) + f^2(b)],$$

$$V_2 = \frac{\pi(b-a)}{3}\left[3f^2\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{4}\left(f'\left(\frac{a+b}{2}\right)\right)^2\right].$$

Since, when f is positive and convexe, we have $V \le V_1$, and under certain conditions $V_2 \le V$, one can deduce the following result.

Theorem 2.8. *Let $f : [a, b] \to \mathbb{R}$ be nonnegative and convex. Then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \le \frac{1}{3}[f^2(a) + f(a)f(b) + f^2(b)]. \tag{18}$$

If, in addition f is differentiable in $x_0 := \frac{a+b}{2}$, and the following condition is satisfied:

(i) $f\left(\frac{a+b}{2}\right) - \frac{b-a}{2}f'\left(\frac{a+b}{2}\right) > 0$ *and* $f'\left(\frac{a+b}{2}\right) > 0$, *then*

$$\frac{1}{b-a}\int_a^b f^2(x)dx \ge f^2\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{12}\left[f'\left(\frac{a+b}{2}\right)\right]^2. \tag{19}$$

Proof. The above stated geometric arguments for the proof of (18) and (19) can be made rigorous. Indeed, for (18), let $K : [a, b] \to \mathbb{R}$ be a linear function having the properties $f(a) = K(a)$, $f(b) = K(b)$. Therefore,

$$K(t) = \frac{t-a}{b-a} f(b) + \frac{b-t}{b-a} f(a), \quad t \in [a, b].$$

Since f is convex and positive, $f^2(t) \le K^2(t)$. Since it is immediate that

$$\int_a^b K^2(t)dt = \frac{b-a}{3}[f^2(a) + f(a)f(b) + f^2(b)],$$

the result follows. For the proof of (19) let us remark that $f(x) \ge f(x_0)+(x-x_0)f'(x_0)$ for all $x \in [a, b]$, $x_0 \in (a, b)$. Put $x_0 := \frac{a+b}{2}$ and write that

$$f^2(x) \ge \left[f(x_0) + \left(x - \frac{a+b}{2}\right) f'(x_0)\right]^2,$$

where $f'(x_0) > 0$. An elementary computation shows that

$$\int_a^b \left[f(x_0) + \left(x - \frac{a+b}{2}\right) f'(x_0)\right]^2 dx = f^2\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{12}\left[f'\left(\frac{a+b}{2}\right)\right]^2,$$

and this finishes the proof.

Remark 2.4. Without differentiability one can assume only that

$$f'_+\left(\frac{a+b}{2}\right) > 0 \quad \text{and} \quad f\left(\frac{a+b}{2}\right) - \frac{b-a}{2} f'_+\left(\frac{a+b}{2}\right) > 0.$$

When f is nonnegative differentiable, and concave, without any condition one has

$$\frac{1}{b-a}\int_a^b f^2(x)dx \le f^2\left(\frac{a+b}{2}\right) + \frac{(b-a)^2}{12}\left[f'\left(\frac{a+b}{2}\right)\right]^2 \tag{20}$$

Indeed, by $0 < f(x) \le f(x_0) + f'(x_0)(x - x_0)$, by taking squares and integrating, we obtain (20). Without differentiablity (20) holds with $f'_+\left(\frac{a+b}{2}\right)$ in place of $f'\left(\frac{a+b}{2}\right)$.

Since $\frac{x^2+xy+y^2}{3} \le \frac{x^2+y^2}{2}$, inequality (18) refines the right side of Hadamard's inequality applied to the convex function f^2. Inequality (18) has been applied in the Theory of means ([14]).

Let $p : [a, b] \to \mathbb{R}$ be a strictly positive monotone function, and define

$$E_{p,f}(a, b) = E_{p,f} = \int_a^b p(x)f(x)dx \Big/ \int_a^b p(x)dx.$$

In paper [20] the following results have been proved:

Theorem 2.9. *Let f be a convex function. Then*

$$E_{p,f} \geq f(A) + f'_+(A)C_p, \tag{21}$$

where $A = \dfrac{a+b}{2}$, *and* $C_p = C_p(a,b)$. *If p is increasing, the $C_p \geq 0$; while for decreasing p one has $C_p \leq 0$.*

Remark 2.5. Therefore, when $f'_+(A) \geq 0$ one can deduce

$$E_{p,f} \geq f(A) + f'_+(A)C_p \geq f(A), \text{ for increasing } p.$$

This generalizes and refines the left side of Hadamard's inequality.

Theorem 2.10. *Let f be convex, with $f(b) \geq f(a)$. If p is a decreasing function, then*

$$E_{p,f} \leq f(a) + \frac{f(b)-f(a)}{b-a}\int_a^b (x-a)p(x)dx \leq \frac{f(a)+f(b)}{2} \tag{22}$$

The same is valid if $f(b) \leq f(a)$ and p increasing.

Finally, as a generalization of (18) we quote (see [20]):

Theorem 2.11. *Let f be positive and convex, with $f(b) \geq f(a)$. Let p be a decreasing function. Then*

$$E_{p,f^n} \leq \sum_{k=0}^{n}\binom{n}{k} f^k(a)\left(\frac{f(b)-f(a)}{b-a}\right)^{n-k}\int_a^b (x-a)^{n-k}p(x)dx \leq$$

$$\leq \sum_{k=0}^{n}\binom{n}{k}\frac{f^k(a)(f(b)-f(a))^{n-k}}{n-k+1} \tag{23}$$

(Here $n \geq 1$ is a positive integer and $\binom{n}{k}$ denotes a binomial coefficient.)

Theorem 2.12. *Let f be positive and concave. Then*

$$E_{p,f^n} \leq \sum_{k=0}^{n}\binom{n}{k} f^{n-k}(A)(f'_+(A))^k\int_a^b (x-A)^k p(x)dx. \tag{24}$$

If $f'_+(A) \geq 0$ and p is decreasing, the right side of (24) can be majored by

$$\sum_{k=0}^{n}\binom{n}{k} f^{n-k}(A)(f'_+(A))^k(b-a)^k\frac{1+(-1)^k}{(k+1)\cdot 2^{k+1}}. \tag{25}$$

Remark 2.6. For $p \equiv 1$, $n=2$, f positive and *concave* one obtains the inequality

$$\frac{1}{b-a}\int_a^b f^2(x)dx \leq f^2(A) + (f'_+(A))^2\frac{(b-a)^2}{12}. \tag{26}$$

C. In the precedent paragraph in certain cases we have obtained refinements of the Hadamard inequality (or for one part of it).

Let now suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has a strictly increasing derivative on (a, b). By Lagrange's mean-value theorem easily follows $f(x) - f(y) < f'(x)(x - y)$ for all $x, y \in (a, b)$. By integrating with respect to x we get

$$\int_a^b f(x)dx < (b-a)f(y) - y[f(b) - f(a)] + \lambda = g(y),$$

where

$$\lambda = \int_a^b xf'(x)dx = bf(b) - af(a) - \int_a^b f(x)dx$$

and $g : [a, b] \to \mathbb{R}$ is defined as above. Clearly, $g'(y) = (b-a)f'(y) - [f(b) - f(a)]$, so by the Lagrange mean-value theorem, $g'(y_0) = 0$ for some $y_0 \in (a, b)$. Since f' is strictly increasing, obviously $g'(y) > g'(y_0) = 0$ for $y > y_0$ and $g'(y) < g'(y_0) = 0$ for $y < y_0$. Therefore y_0 is a minimum-point of the function g, that is $g(y_0) \leq g(y)$ for all $y \in [a, b]$. Thus we have obtained the following result, which in fact appeared in [19]:

Theorem 2.13. *If f satisfies the above conditions, then*

$$\int_a^b f(x)dx < \frac{b-a}{2}\left\{f(y_0) - y_0\left[\frac{f(b)-f(a)}{b-a}\right] + \frac{bf(b)-af(a)}{b-a}\right\} \tag{27}$$

where y_0 is defined by the equality

$$f'(y_0) = \frac{f(b)-f(a)}{b-a}. \tag{28}$$

For this choice of $y = y_0$, inequality (27) is optimal.

Remark 2.7. Clearly, inequality (27) is valid for all $y_0 \in (a, b)$, but for y_0 given by (28) we obtain the strongest result. By selecting $y_0 = \dfrac{a+b}{2}$ in (27) we get the following refinement of the right side of Hadamard's inequality:

$$\int_a^b f(x)dx < \frac{b-a}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] < \frac{b-a}{2}[f(a)+f(b)]. \tag{29}$$

This is due to P.S. Bullen [7].

Indeed, the first inequality is a consequence of (7), while the second one is equivalent to $f\left(\dfrac{a+b}{2}\right) < \dfrac{f(a)+f(b)}{2}$.

Remark 2.8. Inequality (27) is valid also for a strictly convex function f, and has been rediscovered in [3]. For applications in the theory of means, see [19], [3].

The following refinements of the Hadamard inequalities have been published by the author in cooperation with J.E. Pečarić and S.S. Dragomir [15]:

Theorem 2.14. *Let $n \geq 1$ be a positive integer and let $f : [a,b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\displaystyle\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n+1} \frac{x_i}{n+1}\right) dx_1 \ldots dx_{n+1}}{(b-a)^{n+1}} \leq$$

$$\leq \frac{\displaystyle\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^{n} \frac{x_i}{n}\right) dx_1 \ldots dx_n}{(b-a)^n} \leq \cdots \leq \frac{\displaystyle\int_a^b \int_a^b f\left(\frac{x_1+x_2}{2}\right) dx_1 dx_2}{(b-a)^2} \leq$$

$$\leq \frac{\displaystyle\int_a^b f(x)dx}{b-a} \leq \frac{f(a)+f(b)}{2}. \tag{30}$$

Remark 2.9. When $n = 1$ we have the following simple relations:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^2} \iint\limits_{[a,b]^2} f\left(\frac{x+y}{2}\right) dxdy \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{31}$$

In applications (e.g. in the theory of Euler Gamma function), this inequality has a special importance.

D. We will conclude our survey with the study of certain mappings associated to the Hadamard inequalities.

Let $f : [a,b] \to \mathbb{R}$ be a convex function, and define the following mappings: $H, G, L : [0,1] \to \mathbb{R}$, given by

$$H(t) = \frac{1}{b-a}\int_a^b f\left[tx + (1-t)\frac{a+b}{2}\right] dx, \tag{32}$$

$$G(t) = \frac{1}{2}\left\{f\left[ta + (1-t)\frac{a+b}{2}\right] + f\left[(1-t)\frac{a+b}{2} + tb\right]\right\}, \tag{33}$$

$$L(t) = \frac{1}{2(b-a)}\int_a^b \{f[ta + (1-t)x] + f[(1-t)x + tb]\}dx. \tag{34}$$

The following three theorems contain certain properties of these mappings (see [18]).

Theorem 2.15. *Let H be defined by (32). Then*

(i) $$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \leq \int_0^1 H(t)dt \leq$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx\right] \tag{35}$$

and H is a convex mapping.

(ii) If f is differentiable (and convex), then

$$0 \leq \frac{1}{b-a}\int_a^b f(t)dt - H(t) \leq (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right] \tag{36}$$

and

$$0 \leq \frac{f(a)+f(b)}{2} - H(t) \leq \frac{[f'(b)-f'(a)](b-a)}{4}, \quad t \in [0,1]. \tag{37}$$

Remark 2.10. Relation (36) gives a new refinement of the right side of (1).

Theorem 2.16. *Let G be defined by (33). Then*

(i) G is convex and increasing on $[0,1]$;

(ii) $\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$; $\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2}$;

(iii) $H(t) \leq G(t)$ for all $t \in [0,1]$;

(iv) $$\frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(x)dx \leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \leq \int_0^1 G(t)dt \leq$$
$$\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right];$$

(v) If f is differentiable (and convex), then

$$0 \leq H(t) - f\left(\frac{a+b}{2}\right) \leq G(t) - H(t) \text{ for all } t \in [0,1].$$

Remark 2.11. Since $H(1) = \frac{1}{b-a}\int_a^b f(x)dx$, (iii) gives a generalization, while (v) a refinement of Hadamard's inequalities.

Theorem 2.17. *Let L be defined by (34). Then*

(i) L is a convex mapping on $[0,1]$;

(ii) $G(t) \leq L(t) \leq \frac{1-t}{b-a}\int_a^b f(x)dx + t\frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2}$ *for all* $t \in [0,1]$; *and* $\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}$;

(iii) $H(1-t) \leq L(t)$ *and* $\frac{H(t)+H(1-t)}{2} \leq L(t)$ *for all* $t \in [0,1]$.

Remark 2.12. Since $L(0) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$, relation (ii) offers a generalization and new refinement of Hadamard's inequalities.

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Note added in proof. In the first part ([23]) for Theorem 1.3 the Reference [29] is stated incorrectly. The paper in question is the following: **J.E. Pečarić**, Remark on an inequality of S. Gabler, J. Math. Anal. Appl. **184**(1994), 19-21.

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