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HYDRODYNAMICAL CONSIDERATIONS ON THE GAS STREAM IN THE LAGRANGIAN POINT L_1 OF A CLOSE BINARY SYSTEM

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Abstract. Taking into consideration Euler's equation for an ideal fluid and having in view some basic hypotheses specified for hydrodynamic approach, in the case when the potential of massic forces is of Roche type, an integral of Bernoulli type is established. It is shown that it is impossible for a fluid to surpass a certain maximum velocity v_{max} and that the critical velocity of the sound in a fluid depends on the parameters of the Roche model. The conditions in which the fluid motion is subsonic or supersonic are analyzed. In addition the density, the pressure of the fluid and the sound velocity are expressed as function of the fluid velocity and the potential of the massic forces. Then, the Lagrangian point L_1 is considered as a source with its output q and the fluid motion is analyzed in the corresponding close vicinity. The obtained results could also be used as initial conditions for the integration of the mass transfer equations.

1. **Introduction**

The problem of the mass transfer in stellar binary systems is relatively old. It was approached by Kuiper (1941) in his pioneer work. Then, Kopal (1958), Kruszewsky (1964), Plavec et al. (1964, 1965) and many other authors have reviewed this problem, especially with considerations on the orbital period changes. In the above-mentioned papers, the problem of the mass transfer was approached without any hydrodynamic considerations. With computed orbits of single particles, some of the above mentioned authors have demonstrated, theoretically, the existence of rings around the primary component, or some gas streams in the corresponding systems. Other authors also tried to use hydrodynamical considerations: Prendergast (1960), Biermann (1971), Prendergast and Taam (1974). Nevertheless, it was used an arbitrary way to establish the initial conditions for the integration of the differential equations of the correspondent motion. That is why, by taking into consideration some basic hypotheses we have studied the problem of the mass transfer in a close vicinity of the inner Lagrangian point, L_1 and the corresponding results could be used for a better estimation of the initial conditions.

2. **Basic hypotheses**

For the study of the mass transfer in close binary systems, through the methods appropriate to the hydrodynamics, we are obliged to make some hypotheses in order to draw the theoretical model very near to the physical reality. These basic hypotheses will be reviewed in the present section:

a) The two component stars of a binary systems are revolving in circular orbits, about their common mass center. Such an approximation is suited for a great majority of the close binary systems.

b) The fluid flow is assumed as being stationary. This hypothesis is good enough for the detached and semi-detached binary systems, whose light curves have the same behaviour in each cycle, with some small irregularities. Such an assumption is not suited for those binary systems whose stellar components are in contact and the corresponding irregularities are very frequent and well marked.

c) The gas flow is considered only in the orbital plane and an approach of the two dimensions problem may be accepted. This assumption is based on the fact that the resultant force of the corresponding effective forces lies in the orbital plane and it has an endeavour to press the gas stream towards this plane. Indeed, the effective forces, which are operating on the gas stream, are: the forces of the gravitational attraction , the centrifugal force and the Coriolis force . Here we have in view a rotating barycentric coordinates system (M,x,y,z) where the origin M is situated in the common mass-center and the two component stars S_1 and S_2 are always situated on the x-axis, while (x,M,y) plane coincides with the orbital plane. In such conditions the gravitational forces are given by:

$$\overrightarrow{F}_{atr_1} = -G\frac{m_1 m}{r_1^2}\frac{\overrightarrow{r}_1}{r_1}, \qquad \overrightarrow{F}_{atr_2} = -G\frac{m_2 m}{r_2^2}\frac{\overrightarrow{r}_2}{r_2}, \tag{1}$$

with: $\overrightarrow{r}_1 = (x + R_1)\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$, $\overrightarrow{r}_2 = (x - R_2)\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$,
where R_1 and R_2 are the distances of the two stellar components from the common mass-center.

In addition we have:

$$\overrightarrow{F}_{centrif} = -m\overrightarrow{\omega} \times (\overrightarrow{\omega} \times \overrightarrow{r}) = m\omega^2 \left(x\,\overrightarrow{i} + y\,\overrightarrow{j} \right) \quad (2)$$

$$\overrightarrow{F}_{Coriolis} = -2m\overrightarrow{\omega} \times \overrightarrow{v_r} = 2m\omega \left(y\,\overrightarrow{i} - x\,\overrightarrow{j} \right) \quad (3)$$

with: $\overrightarrow{v}_r = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$.

In Eqs. (1) - (3) $m_{1,2}$ are the masses of the two stellar components (S_1 and S_2), while m represents the mass of a gas particle. As it is shown by Biermann (1971), the use of two dimensions only (in the orbital plane) is equivalent either to a cylindrical model of the system or, to a gas flow of constant thickness. The corresponding thickness may be considered as a function of temperature.

d) The gas flow is assumed as being adiabatic. Such an assumption is suited if:

- the mean free path for photons is small compared to the characteristic dimensions of the considered binary - system;

- the thermal time scale of the gas is long compared to the transfer time scale.

Now, from theoretical considerations on the mass transfer in close binary - systems (eg. Kippenhahn et al., 1967) it follows the fact that the corresponding problem has two phases: the first one is characterized by a fast flow of the gas, to the thermal time scale of the star which is losing mass. The second phase is characterized by a slow gas flow, to the nuclear time scale. In the slow phase of the mass transfer, the corresponding gases may be considered as being transparent and their thermal behaviour is determined by the radiation field of the two stellar components .

In the phase of fast mass transfer, the mean free path for photons is small enough while the thermal time scale is long enough in order to surmise that adiabacy is a good approximation.

e) We are considering a binary system as being semi-detached because the assumptions b) and d) are not suited for contact systems.

f) We are assuming that for pressure the law of perfect gas may be adopted:

$$P = \frac{k\varrho T}{\mu m_a} \tag{4}$$

where ϱ is the fluid density, T is the temperature, $k = 1,38054 * 10^{-23} JK^{-1}$ is Boltzmann‘s, constant, μ is the relative molecular mass (in units of atomic mass) and m_a is the proton mass (see Ureche, 1987).

g) The gas flow is assumed as being laminar throughout, without turbulence and irrotational . As it is known, (Biermann, 1971), the Reynolds number can be written as the product of the Mach number of the flow and ratio of a characteristic length scale to the gas-dynamical mean free path. The Reynolds number for the gas flow in binary systems is evaluated as being of the order of 10^8 (Kopal,1958). On the other hand it is known from experiments (e.g. Biermann, 1971) that the supersonic turbulence is strong connected with the properties of the boundary layers. Nevertheless, there are no fixed boundaries in a close binary system and, consequently, there is no a simple theory to discuss the properties of possible boundary layers. In such conditions it is very difficult to draw an important conclusion concerning the turbulence. For simplicity the gas flow is assumed as being laminar throughout.(obvious with the exception of the area situated behind the shock wave) .

h) The magnetic fields are neglected, even if they could be important in some binary systems. But, as it is shown (Biermann, 1971) in the phase of fast mass transfer, the magnetic fields are important only if their strength is of the order of 10^3 Gauss or greater . In the phase of slow mass transfer 10 Gauss have already appreciable effects. But for the phase of fast mass transfer no observed example is known to have such a magnetic field. On the other hand, for the phase of slow mass transfer which can be identified with many observed binary systems, the value of 10 Gauss is below observational limits. Anyhow, if we take them into consideration, the problem becomes more complicated, because in the motion equations we have to add a supplemental term of form $\overrightarrow{j} \times \overrightarrow{B}$, where $\overrightarrow{j}$ is the density of the stream while $\overrightarrow{B}$ is the magnetic induction. In addition, in equation of the energy we should have a supplemental term of the form $\overrightarrow{j} \cdot \overrightarrow{E}$, where $\overrightarrow{E}$ represents the strength of the electric field (Ureche, 1987).

i) For the gravitational field of each stellar component, the Roche potential is assumed. Since the stars, which are losing mass, are evolving far from main sequence, they have an increased concentration of density. Therefore, the tides do not change very much the gravitational potential. Hence, the Roche potential may be considered as a good approximation.

j) Finally, we are assuming that there is a synchronization between the axial rotation of the two stellar components and the corresponding orbital motion, that is, for the angular velocity, we can write: $\omega = \frac{2\pi}{P} = const.$

3. Subsonic, supersonic and hypersonic motions in the jet of stellar matter

From the Euler's equation, written for an ideal fluid, that is:

$$\varrho \left[\frac{\partial \overrightarrow{v}}{\partial t} + \frac{1}{2} grad \left(v^2 \right) + rot\, \overrightarrow{v} \times \overrightarrow{v} \right] = \varrho \overrightarrow{f} - grad\, P,$$

in the hypothesis of a steady state, we can write:

$$\varrho \left[\frac{1}{2} grad \left(v^2 \right) + rot\, \overrightarrow{v} \times \overrightarrow{v} \right] = \varrho \overrightarrow{f} - grad\, P. \tag{5}$$

If we are considering now that there is a scalar function U (potential) so that

$$\overrightarrow{f} = grad\, U,$$

then Eq.(5) can be written in the form:

$$\varrho \left[\frac{1}{2} grad \left(v^2 \right) + rot\, \overrightarrow{v} \times \overrightarrow{v} \right] = \varrho\, grad\, U - grad\, P. \tag{6}$$

In addition, if we assume that the compressible fluid is a barotropic one, we may consider that there is a scalar function h , so that $grad\, h = \frac{grad\, P}{\varrho}$. Therefore, from Eq. (6) it is evident that on any stream line we have:

$$\frac{v^2}{2} - U + h = C_1 = const. \tag{7}$$

which, in fact, is the Bernoulli's integral.

In the hypothesis of a perfect gas and in an izentropic evolution, we can write: $P = k\, \varrho^{\gamma}$, where $\gamma > 1$ represents the adiabatic exponent.

Moreover, from the relationship: $h = \int dh = \int \frac{dP}{\varrho}$ we have at once:

$$h = \frac{\gamma}{\gamma - 1} \frac{P}{\varrho}, \tag{8}$$

and the Hugoniot formula leads to:

$$c^2 = \frac{dP}{d\varrho} = \gamma \frac{P}{\varrho}. \tag{9}$$

Therefore Eqs. (8) and (9) lead to:

$$h = \frac{c^2}{\gamma - 1} \tag{10}$$

In such conditions, Eq. (7) becomes:

$$\frac{v^2}{2} - U + \frac{c^2}{\gamma - 1} = C_1 \,. \tag{11}$$

Here the constant C_1 may be determined if we use Eq. (11) for an arbitrary point situated somewhere on the Roche equipotential surface. On such a surface the velocity is null and the corresponding potential is $U_{Roche} = const.$ On the other hand, it is known that on such equipotential surfaces, we also have a constant density, hence $\varrho_{Roche} = const.$

Now, from relationship $P = k\,\varrho^{\gamma}$, written for an arbitrary point of the Roche equipotential surface, we have:

$$P_{Roche} = k\,\varrho_{Roche}^{\gamma} = constant$$

and Eq. (9) leads to:

$$c_{Roche}^2 = \gamma \frac{P_{Roche}}{\varrho_{Roche}} = constant.$$

For an arbitrary point of the Roche equipotential surface we are able to determine the value of the constant C_1 , that is:

$$-U_{Roche} + \frac{c^2}{\gamma - 1} = C_1.$$

and Eq. (11) may be written in the form:

$$\frac{v^2}{2} - U + \frac{c^2}{\gamma - 1} = -U_{Roche} + \frac{c_{Roche}^2}{\gamma - 1} \tag{12}$$

whence we have at once:

$$c^2 = c_{Roche}^2 \left[1 - \frac{(\gamma - 1)(v^2 - 2U + 2U_{Roche})}{2c_{Roche}^2} \right]. \tag{13}$$

The solution of Eq.(13) will be found in the range of real numbers only if it is satisfied the condition:

$$1 \geq \frac{(\gamma - 1)(v^2 - 2U + 2U_{Roche})}{2c_{Roche}^2},$$

or:

$$v^2 \leq \frac{2c_{Roche}^2}{\gamma - 1} + 2\left(U - U_{Roche}\right).$$

Therefore, in its motion, a fluid cannot surpass (exceed) a maximum velocity v_{max} , given by the relationship:

$$v_{max}^2 = \frac{2c_{Roche}^2}{\gamma - 1} + 2\left(U - U_{Roche}\right). \tag{14}$$

If in a point situated somewhere on a stream line, the fluid velocity becomes equal to the sound velocity in the same point, that is if we can write $v = c = c_*$, then from Eq. (12) we have:

$$c_*^2 = \frac{\gamma - 1}{\gamma + 1}\left[2\frac{c_{Roche}^2}{\gamma - 1} + 2\left(U_* - U_{Roche}\right)\right].$$

where c_* represents the critical sound velocity in fluid, while U_* is the corresponding potential.

If $v > c_*$ we have the case of the supersonic motion.

If $v < c_*$ we have the case of the subsonic motion. Furthermore, from Eq.(9) we can write: $c^2 = k\gamma\varrho^{\gamma-1}$ or $c_{Roche}^2 = k\gamma\varrho_{Roche}^{\gamma-1}$ and Eq.(13) becomes:

$$\varrho = \varrho_{Roche}\left[1 - \frac{(\gamma - 1)\left(v^2 - 2U + 2U_{Roche}\right)}{2c_{Roche}^2}\right]^{\frac{1}{\gamma-1}}. \tag{15}$$

Here we have in view the relationship: $P = k\varrho^{\gamma}$ or: $\varrho = \left(\frac{P}{k}\right)^{\frac{1}{\gamma}}$ and Eq. (8) leads to:

$$P = P_{Roche}\left[1 - \frac{(\gamma - 1)(v^2 - 2U + 2U_{Roche})}{2c_{Roche}^2}\right]^{\frac{\gamma}{\gamma-1}}. \tag{16}$$

In conclusion, the relationships (13), (15) and (16) give us the explicit functions c(v), $\varrho(v)$ and P(v). During the study of the mass transfer, it was put in evidence a very luminous patch - a hot spot - in that place where the jet of matter hit the atmosphere of the star which is receiving mass. The existence of such a patch was also detected by observational methods. Consequently it was created a true theory of such named "hot spot", in order to explain some irregularities (fluctuations) observed in the light curves of the eclipsing binary systems. Prendergast and Taam (1974) try to explain such a hot spot as a consequence of the heating determined by the shock wave which arise in that place and have estimated a temperature of the order of 35000 K. In front of the shock wave the jet motion is hypersonic, that is the jet matter must be accelerated by the gravitational attraction until to velocities characterized by the

Mach number $M \geq 5$. If we assume that the fluid motion is *hypersonic*, *izentropic* and *steady* (the jet matter being assumed as a perfect gas), for each stream line we have dS = 0. Further, from Eq. (7) we have at once:

$$v\,dv - dU + dh = 0$$

but $dh = \frac{dP}{\varrho}$ and consequently we can write:

$$\frac{dP}{P} = \frac{\varrho}{P}dU - \frac{\varrho v}{P}dv.$$

Here we can use the following relationship: $\frac{\varrho}{P} = \frac{\gamma}{c^2}$ and consequently we have:

$$\frac{dP}{P} = \frac{\gamma}{c^2}dU - \frac{\gamma v}{c^2}dv$$

Finally, if we have in view the Mach number $M = \frac{v}{c}$, we can write :

$$\frac{dP}{P} = \frac{\gamma M^2}{v^2}dU - \gamma M^2\,\frac{dv}{v}. \tag{17}$$

As it was before mentioned, we can use the relationship: $c^2 = \frac{\gamma P}{\varrho}$ and, if we accept Clapeyron law: $P = \varrho \mathcal{R} T$ it follows that:

$$c^2 = \gamma \mathcal{R} T \tag{18}$$

or, by differentiation it follows that:

$$2c\,dc = \gamma dT\mathcal{R} \tag{19}$$

Now, from Eqs. (18) and (19) we have at once:

$$\frac{dT}{T} = 2\frac{dc}{c}, \tag{20}$$

and from Eq.(11) on a stream line we can write:

$$v\,dv - dU + \frac{2c\,dc}{\gamma - 1} = 0. \tag{21}$$

In such conditions, Eq. (20) becomes:

$$\frac{dT}{T} = -\frac{(v\,dv - dU)\,(\gamma-1)}{c^2}$$

or, if we have in view the Mach number, M:

$$\frac{dT}{T} = -(\gamma-1)\,M^2\left(\frac{dv}{v} - \frac{dU}{v^2}\right). \tag{22}$$

From Eqs. (17) and (22) it is evident that, since M^2 is a great number (the fluid motion being assumed as hypersonic one) , to a small change in the velocity on a stream line could correspond a great change for the pressure and temperature. From the relationship: $M = \frac{v}{c}$ we can write:

$$\frac{dM}{M} = \frac{dv}{v}\left(1 - M\,\frac{dc}{dv}\right) \tag{23}$$

By differentiation, from Eq. (11) it follows:

$$v - \frac{dU}{dv} + \frac{2c}{\gamma - 1}\,\frac{dc}{dv} = 0$$

and Eq.(23) can be written in the form:

$$\frac{dM}{M} = \frac{dv}{v\left[1 - M\frac{\gamma-1}{2c}\left(-v + \frac{dU}{dv}\right)\right]},$$

or

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2 - M\,\frac{\gamma - 1}{2c}\,\frac{dU}{dv}\right)\frac{dv}{v}. \tag{24}$$

Moreover, from Eq.(21) we can obtain a relationship for dv , and Eq.(24) becomes:

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2 - M\,\frac{\gamma - 1}{2c}\,\frac{dU}{dv}\right)\left[\frac{dU}{v^2} - \frac{2}{M^2(\gamma - 1)}\,\frac{dc}{c}\right]. \tag{25}$$

If we consider the fluid motion as being a hypersonic one, we can use the following approximation:

$$\frac{dU}{v^2} \approx 0.$$

and Eqs. (17), (22) and (25) lead to:

$$\frac{dP}{P} = -\gamma M^2\,\frac{dv}{v} \tag{26}$$

$$\frac{dT}{T} = -(\gamma - 1)M^2\,\frac{dv}{v} \tag{27}$$

$$\frac{dM}{M} = \left(1 + \frac{\gamma - 1}{2}M^2 - M\frac{\gamma - 1}{2c}\,\frac{dU}{dv}\right)\left[-\frac{2}{M^2(\gamma - 1)}\,\frac{dc}{c}\right]. \tag{28}$$

The shock wave, which arises in the close vicinity of the secondary component, practically is stuck to this star. The very high temperatures from behind of the shock determine ionization and dissociation of the particles and, consequently, the laminar model of the fluid cannot be used. Behind of the shock wave arises a zone of turbulence that, in fact, is a zone of the complementarity of the secondary star. Moreover, as a strong increasing of the temperature, this zone can be put in evidence through the direct astronomical observations.

4. The study of the fluid motion in the close vicinity of the point L_1

Further, in the present section we shall consider that in the Lagrangian point L_1 we have a mass source, with the corresponding output q . In such a condition, the continuity equation (see L. Dragos, 1981) will be written in the form:

$$div\,(\varrho\,\overrightarrow{v}) = q\,\delta\,(\overrightarrow{x})\,, \tag{29}$$

where $\delta\,(\overrightarrow{x})$ is Dirac‘s distribution. Let us consider that the fluid motion take place in the orbital plane, where $r = |\overrightarrow{x}|$. Thus, from the study of the Roche equipotentials it is known that the tangents, in the orbital plane, drawn in L_1, have the corresponding slopes θ_0 and $-\theta_0$, where:

$$tg^2\theta_0 = \frac{2x_{L_1}^{-3} + 2\frac{m_2}{m_1}\left(1 - x_{L_1}\right)^{-3} + \left(1 + \frac{m_2}{m_1}\right)}{x_{L_1}^{-3} + \frac{m_2}{m_1}\left(1 - x_{L_1}\right)^{-3} - \left(1 + \frac{m_2}{m_1}\right)} \tag{30}$$

(The corresponding numerical values are listed by Plavec and Kratochvil (1964), for the mass ratio $\frac{m_2}{m_1}$).

In the hypothesis that in the range $\theta_0 \in [-\theta_0, \theta_0]$ there are not preferential directions, Eq. (29) becomes (see L. Dragos, 1981):

$$\frac{1}{r}\,\frac{d}{dr}(\varrho r v) = \frac{q}{2\pi r}\,\delta(r). \tag{31}$$

The corresponding homogenous equation of Eq.(31) is:

$$\frac{1}{r}\,\frac{d}{dr}(\varrho r v) = 0,$$

which have the solution:

$$\varrho v = \frac{C}{r}. \tag{32}$$

The value of the constant C will be determined in such a way that the equation of the continuity to be total satisfied, and not only in a certain range which do not contain the origin and is specified by the relationship: $C = \frac{q}{2\pi}$. In such a case, the solution (32) becomes:

$$\varrho v = \frac{q}{2\pi r} \tag{33}$$

with $r^2 = x^2 + y^2$. From the relationships: $c^2 = \frac{dP}{d\varrho}$ and $P = k\varrho^\gamma$ we obtain $\varrho = \left(\frac{c^2}{k\gamma}\right)^{\frac{1}{\gamma-1}}$ and from Eq. (33) we have at once:

$$v = \frac{q}{2\pi(k\gamma)^{\frac{1}{1-\gamma}}} \frac{c^{\frac{2}{1-\gamma}}}{r}. \tag{34}$$

Now, from Eq. (12) we have:

$$c^2 = (\gamma - 1)\left(U - U_{Roche} + \frac{c^2_{Roche}}{\gamma - 1} - \frac{v^2}{2}\right),$$

and Eq.(34) can be written in the form:

$$v = \frac{q}{2\pi(k\gamma)^{\frac{1}{1-\gamma}}} \frac{(\gamma-1)^{\frac{1}{1-\gamma}}}{r} \left(U - U_{Roche} + \frac{c^2_{Roche}}{\gamma - 1} - \frac{v^2}{2}\right)^{\frac{1}{1-\gamma}}. \tag{35}$$

Therefore if we consider now a point $A\,(x_A, y_A)$ on a stream line, we know the value U_A and $r_A^2 = x_A^2 + y_A^2$, and from Eq.(35) we obtain the value of v_A .

If $v_A > c_*$, we have a supersonic motion.

If $v_A < c_*$ the motion is subsonic.

Finally, Eq. (35) could also be used in order to obtain the initial value of the velocity, which is useful in order to perform the integration of the equation of fluid motion at a great distance from L_1, but taking a suitable value for r . That is why, the study of the fluid motion could be performed on a natural way, the initial conditions being not imposed in an arbitrary way.

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