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ON THE CONVERGENCE OF ITERATIVE PROCESS FOR NON-LOCAL PROBLEM SOLUTION IN SURGERY

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Abstract. In this study a priori estimate for the solution of parabolic differential equation in surgery was found. Then this estimate was used to prove the convergence of iterative process.

1. In domain $Q = \Omega x\,(0,T)\,, \Omega \equiv \{x = (x_1, x_2) : r_0 < x_1 < R,\ 0 < x_2 < \pi\}$, consider the following problem

$$\frac{\partial u}{\partial t} = \frac{1}{x_1}\frac{\partial}{\partial x_1}\left(x_1 k\,(x,t)\,\frac{\partial u}{\partial x_1}\right) + \frac{1}{x_1^2}\frac{\partial}{\partial x_2}\left(k\,(x,t)\,\frac{\partial u}{\partial x_2}\right) + f\,(x,t)\,, \tag{1}$$

$$\begin{cases} k\frac{\partial u}{\partial x_1}|_{x_1=r_0} = \int_{r_0}^{\alpha} u dx_1 - \mu_1\,(t, x_2)\,, & x_1 = r_0, \\ -k\frac{\partial u}{\partial x_1} = \beta u - \mu_2\,(t, x_2)\,, & x_1 = R, \end{cases} \tag{2}$$

$$\begin{cases} k\frac{1}{x_1}\frac{\partial u}{\partial x_2} = \gamma_1 u - \chi_1\,(x_1, t)\,, & x_2 = 0, \\ -k\frac{1}{x_1}\frac{\partial u}{\partial x_2} = \gamma_2 u - \chi_2\,(x_1, t)\,, & x_2 = \pi, \end{cases} \tag{3}$$

$$u\,(x, 0) = u_0\,(x)\,, \tag{4}$$

where α is a certain number of the interval (r_0, R), $k\,(x,t) \geq c_1 > 0, u_0\,(x)\,, f, \beta, \mu_\upsilon, \gamma_\upsilon, \chi_\upsilon,\ (\upsilon = 1,2)$ are well known functions which satisfies smoothness conditions necessary for the solution to exist [1]. Problem (1)-(4) appeared during mathematical simulation of the processes of the heat transference into tissue [5]. In [1], the existence of the solution of the problem (1)-(4) is proved by using the potential method.

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Our nearest goal is to get a priori estimate for the solution of problem (1)-(4) whence particularly uniqueness of the solution will be shown. Then the obtained priori estimate will be used to prove iterative process convergence.

Since the problem with non-local condition (2) generates a non self-conjugate problem and the sign of corresponding differential operator is not defined, the general theory developed for stability and difference scheme cannot be applied to the above mentioned problem. Besides, non-local condition does not allow to use of any disjunction scheme for the solution of two-dimensional problem (1)-(4).

All these difficulties can be overcome, if the following sequence of problems is considered instead of problem (1)-(4).

$$\begin{cases} \frac{\partial \overset{s}{u}}{\partial t} = L\overset{s}{u} + f, L\overset{s}{u} \equiv \frac{1}{x_1}\frac{\partial}{\partial x_1}\left(x_1 k \frac{\partial \overset{s}{u}}{\partial x_1}\right) + \frac{1}{x_1^2}\frac{\partial}{\partial x_2}\left(k\frac{\partial \overset{s}{u}}{\partial x_2}\right), \\ \begin{cases} k\frac{\partial \overset{s}{u}}{\partial x_1} = \int_{r_0}^{\alpha} \overset{s-1}{u}\, dx_1 - \mu_1, & x_1 = r_0, \\ -k\frac{\partial \overset{s}{u}}{\partial x_1} = \beta \overset{s}{u} - \mu_2, & x_1 = R, \end{cases} \\ \begin{cases} k\frac{1}{x_1}\frac{\partial \overset{s}{u}}{\partial x_2} = \gamma_1 \overset{s}{u} - \chi_1, & x_2 = 0, \\ -k\frac{1}{x_1}\frac{\partial \overset{s}{u}}{\partial x_2} = \gamma_2 \overset{s}{u} - \chi_2, & x_2 = \pi, \end{cases} \\ \overset{s}{u}(x,0) = u_0(x), & s = 1,2,... \end{cases} \tag{5}$$

where s is iterative index.

In each iteration, the problem (5) becomes regular, so for the numerical solution of (5), local one-dimensional schemes can be used [2]. Now let's multiply equation (1) with the scalar product $x_1 u$ and integrate by parts, we obtain

$$\frac{1}{\alpha}\frac{\partial}{\partial t}\left(x_1, u^2\right) + \left(x_1, k u_{x_1}^2\right) + \left(\frac{1}{x_1}, k u_{x_2}^2\right) + \int_0^{\pi} R\beta u^2\left(R, x_2, t\right) dx_2 +$$
$$+ \int_0^{\pi} r_0 u\left(r_0, x_2, t\right)\left(\int_{r_0}^{\alpha} u\left(x_1, x_2, t\right) dx_1\right) dx_2 + \int_{r_0}^{R} \gamma_2 u^2\left(x_1, \pi, t\right) dx_1 +$$
$$+ \int_{r_0}^{R} \gamma_1 u^2\left(x_1, 0, t\right) dx_1 = (f, x_1 u) + \int_0^{\pi} R\mu_2 u\left(R, x_2, t\right) dx_2 + \int_0^{\pi} r_0 \mu_1 u\left(r_0, x_2, t\right) dx_2 +$$
$$+ \int_{r_0}^{R} \chi_2\left(x_1, t\right) u\left(x_1, \pi, t\right) dx_1 + \int_{r_0}^{R} \chi_1\left(x_1, t\right) u\left(x_1, 0, t\right) dx_1. \tag{6}$$

Let's estimate the right hand-side integrals of (6). Using S.L. Sobolev's embedding theorem [3], we get

$$\int_0^{\pi} r_0\mu_1 u\left(r_0,x_2,t\right)dx_2 \le \frac{r_0}{2}\left(\int_0^{\pi} u^2\left(r_0,x_2,t\right)dx_2 + \int_0^{\pi}\mu_1^2 dx_2\right) \le$$
$$\le \frac{r_0\varepsilon}{2}\int_0^{\pi}\|u_{x_1}\|^2_{L_2(x_1)}dx_2 + \frac{r_0 C_\varepsilon}{2}\int_0^{\pi}\|u_{x_1}\|_{L_2(x_1)}dx_2 + \frac{r_0}{2}\int_0^{\pi}\mu_1^2 dx_2 \le$$
$$\le \frac{\varepsilon}{2}\|\sqrt{x_1}u_{x_1}\|_0^2 + \frac{C_\varepsilon}{2}\|\sqrt{x_1}u\|_0^2 + \frac{r_0}{2}\int_0^{\pi}\mu_1^2 dx_2, \qquad (7)$$

where $\|u\|_{L_2(x_1)}$ means that the norm is taken in correspondence with variable $x_1, \varepsilon > 0, C_\varepsilon > 0$ are positive constants.

In the same way we find the following

$$\int_0^{\pi} R\mu_2 u\left(R,x_2,t\right)dx_2 \le \frac{R}{2r_0}\varepsilon\|\sqrt{x_1}u_{x_1}\|_0^2 + \frac{R}{2r_0}C_\varepsilon\|\sqrt{x_1}u\|_0^2 + \frac{R}{2}\int_0^{\pi}\mu_2^2 dx_2, \qquad (8)$$

$$\int_{r_0}^{R}\chi_2 u\left(x_1,\pi,t\right)dx_1 \le \frac{1}{2}\int_{r_0}^{R}\left(\varepsilon\|u_{x_2}\|^2_{L_2(x_2)} + C_\varepsilon\|u\|^2_{L_2(x_2)}\right)dx_1 + \frac{1}{2}\int_{r_0}^{R}\chi_2^2 dx_1 \le$$
$$\le \frac{R\varepsilon}{2}\left\|\frac{1}{\sqrt{x_1}}u_{x_2}\right\|_0^2 + \frac{C_\varepsilon}{2r_0}\|\sqrt{x_1}u\|_0^2 + \frac{1}{2}\int_{r_0}^{R}\chi_2^2 dx_1, \qquad (9)$$

$$\int_{r_0}^{R}\chi_1 u\left(x_1,0,t\right)dx_1 \le \frac{R\varepsilon}{2}\left\|\frac{1}{\sqrt{x_1}}u_{x_2}\right\|_0^2 + \frac{C_\varepsilon}{2r_0}\|\sqrt{x_1}u\|_0^2 + \frac{1}{2}\int_{r_0}^{R}\chi_1^2 dx_1. \qquad (10)$$

Let's estimate the left hand-side integrals of (6) which corresponds to non-local condition (2):

$$r_0\int_0^{\pi} u\left(r_0,x_2,t\right)\left(\int_{r_0}^{\alpha} u\left(x_1,x_2,t\right)dx_1\right)dx_2 \le$$
$$\le \frac{r_0}{2}\left(\int_0^{\pi}\int_{r_0}^{\alpha} u^2\left(x_1,x_2,t\right)dx_1 dx_2 + \frac{\alpha - r_0}{2}\int_0^{\pi} u^2\left(r_0,x_2,t\right)dx_2\right) \le$$
$$\le \frac{1}{2}\int_0^{\pi}\int_{r_0}^{R} x_1 u^2\left(x_1,x_2,t\right)dx_1 dx_2 + \frac{r_0\left(\alpha - r_0\right)}{2}\int_0^{\pi}\left(\varepsilon\|u_{x_1}\|^2_{L_2(x_1)} + C_\varepsilon\|u\|^2_{L_2(x_1)}\right)dx_2$$
$$\le \frac{1}{2}\left(1 + \left(\alpha - r_0\right)C_\varepsilon\right)\|\sqrt{x_1}u\|_0^2 + \frac{\left(\alpha - r_0\right)}{2}\varepsilon\|\sqrt{x_1}u_{x_1}\|_0^2, \qquad (11)$$

$$(f, x_1 u) \le \frac{1}{2}\|\sqrt{x_1} f\|_0^2 + \frac{1}{2}\|\sqrt{x_1} u\|_0^2, \tag{12}$$

where $\|u\|_0^2 = \int_0^\pi \int_{r_0}^R u^2 dx_1 dx_2$.

Having substituted inequalities (7)-(12) in (6) and chosen sufficiently small ε, we find

$$\frac{\partial}{\partial t}\|\sqrt{x_1} u\|_0^2 + \nu_1 \|\sqrt{x_1} u_{x_1}\|_0^2 + \nu_2 \left\|\frac{1}{\sqrt{x_1}} u_{x_2}\right\|_0^2 \le$$

$$\le \|\sqrt{x_1} f\|_0^2 + M(\varepsilon)\|\sqrt{x_1} u_{x_1}\|_0^2 + R|\mu|^2 + |\chi|^2, \tag{13}$$

where $\nu_1 = 2c_1 - \varepsilon\left[1 + (\alpha - r_0) + \frac{R}{r_0}\right] > 0, \nu_2 = 2(c_1 - R\varepsilon) > 0, M(\varepsilon) = 1 + \left(1 + \frac{2+R}{r_0}\right) C_\varepsilon$,

$$|\mu|^2 = \int_0^\pi (\mu_1^2 + \mu_2^2) dx_2, |\chi|^2 = \int_{r_0}^R (\chi_1^2 + \chi_2^2) dx_1.$$

Integrating the inequality (13) from 0 to t we get

$$\left\|\sqrt{x_1} u\right\|_0^2 + \nu_1 \left\|\sqrt{x_1} u_{x_1}\right\|_{2,Q_t}^2 + \nu_2 \left\|\tfrac{1}{\sqrt{x_1}} u_{x_2}\right\|_{2,Q_t}^2 \le$$

$$\le \|\sqrt{x_1} f\|_{2,Q_t}^2 + M(\varepsilon)\int_0^t \|\sqrt{x_1} u\|_0^2 d\tau + \|\sqrt{x_1} u(x,0)\|_0^2 +$$

$$+R\int_0^t |\mu(\tau)|^2 d\tau + \int_0^t |\chi(\tau)|^2 d\tau, \tag{14}$$

where $\|u\|_{2,Q_t}^2 = \int_0^t \|u\|_0^2 d\tau$.

From inequality (14), we have

$$\|\sqrt{x_1} u\|_0^2 \le M(\varepsilon)\int_0^t \|\sqrt{x_1} u\|_0^2 d\tau + F(t) \tag{15}$$

where $F(t) = \left\|\sqrt{x_1} f\right\|_{2,Q_t}^2 + \left\|\sqrt{x_1} u(x,0)\right\|_0^2 + R\int_0^t |\mu(\tau)|^2 d\tau + \int_0^t |\chi(\tau)|^2 d\tau$.

Using well-known Lemma 1.1 [4], from (15) it is obtained

$$y(t) \le e^{M(\varepsilon)t} tF(t), y(t) = \int_0^t \|\sqrt{x_1} u\|_0^2 d\tau. \tag{16}$$

By making use of (16), we obtain a necessary estimate from (14)

$$\|\sqrt{x_1} u\|_0^2 + \nu_1 \|\sqrt{x_1} u_{x_1}\|_{2,Q_t}^2 + \nu_2 \left\| \frac{1}{\sqrt{x_1}} u_{x_2} \right\|_{2,Q_t}^2 \le$$

$$\le M(t) \left[\|\sqrt{x_1} f\|_{2,Q_t}^2 + \|\sqrt{x_1} u(x,0)\|_0^2 + \int_0^t |\mu(\tau)|^2 d\tau + \int_0^t |\mu(\tau)|^2 d\tau \right]. \tag{17}$$

Since $r_0 < x_1 < R$, $\sqrt{x_1}$ can be ignored in estimate (17). So we have

$$\|u\|_0^2 + \nu_1 \|u_{x_1}\|_{2,Q_t}^2 + \nu_2 \|u_{x_2}\|_{2,Q_t}^2 \le M(t) \left[\|f\|_{2,Q_t}^2 + \|u_0(x)\|_0^2 + \int_0^t \left(|\mu(\tau)|^2 + |\chi(\tau)|^2 \right) d\tau \right]. \tag{18}$$

It is clear from estimate (18) that the problem (1)-(4) has a unique solution.

2. Let's designate $\overset{s}{z} = \overset{s}{u} - u$, then for $\overset{s}{z}$, we have

$$\frac{\partial \overset{s}{z}}{\partial t} = L \overset{s}{z},$$

$$\begin{cases} k \frac{\partial \overset{s}{z}}{\partial x_1} = \int_{r_0}^{\alpha} \overset{s-1}{z} dx_1 \\ -k \frac{\partial \overset{s}{z}}{\partial x_1} = \beta \overset{s}{z}, & x_1 = R, \end{cases}$$

$$\begin{cases} k \frac{1}{x_1} \frac{\partial \overset{s}{z}}{\partial x_2} = \gamma_1 \overset{s}{z}, & x_2 = 0, \\ -k \frac{1}{x_1} \frac{\partial \overset{s}{z}}{\partial x_2} = \gamma_2 \overset{s}{z}, & x_2 = \pi, \end{cases}$$

$$\overset{s}{z}(x,0) = 0.$$

Using estimate (18) for $\overset{s}{z}$, we find

$$\left\| \overset{s}{z} \right\|_0^2 \le M(t) \int_0^t \int_0^\pi \left(\int_{r_0}^{\alpha} \overset{s-1}{z} dx_1 \right)^2 dx_2 dt \le M(T)(\alpha - r_0) \int_0^T \left\| \overset{s-1}{z} \right\|_0^2 dt,$$

or after integration with respect to t from 0 to T , we have

$$\left\|\overset{s}{z}\right\|_{2,Q_T} \leq \sqrt{M(T)\,T\,(\alpha - r_0)}\left\|\overset{s-1}{z}\right\|_{2,Q_T}, M(T) = Te^{M(\varepsilon)T}. \qquad (19)$$

Suppose $q = Te^{M(\varepsilon)T}\sqrt{\alpha - r_0} < 1$. From (19), we obtain the estimate

$$\left\|\overset{s}{z}\right\|_{2,Q_T} \leq q^s \left\|\overset{0}{z}\right\|_{2,Q_T}.$$

Thus the iterative process is convergent in the norm $\|\cdot\|_{2,Q_T}$, for sufficiently small T or small $\alpha - r_0$.

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