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EXPONENTIAL STABILITY OF EVOLUTION OPERATORS

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Abstract. The aim of this paper is to give some sufficient, respectively necessary and sufficient conditions, for the exponential stability of evolution operators in infinite-dimensional spaces. The obtained results are like those, of Datko-type, for evolutionary processes which are linear operators-valued.

1. **Introduction**

Let X be a Banach space and let $(X_t)_{t\geq 0}$ be a family of parts of X.

Definition 1. The family of applications $\Phi(t, t_0) : X_{t_0} \to X_t$, $t \geq t_0 \geq 0$, will be called an evolution operator in X, if the following conditions are satisfied:

i) $\Phi(t, t)x = x$, for all $t \geq 0$ and $x \in X_t$.

ii) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$, for all $t \geq s \geq t_0 \geq 0$.

iii) $\Phi(\cdot, s)x : [s, \infty) \to X$ is continuous, for all $s \geq 0$ and $x \in X_s$.

iv) There is a nondecreasing function $p(\cdot) : \mathbb{R}_+ \to (0, \infty)$, such that

$$\|\Phi(t, s)x\| \leq p(t - s)\|x\|, \text{ for all } t \geq s \geq 0 \text{ and } x \in X_s.$$

Remark 1. Condition iv) can be replaced by

v) There are $M, \omega > 0$ such that

$$\|\Phi(t, s)x\| \leq Me^{\omega(t-s)}\|x\|,$$

for all $t \geq s \geq 0$ and $x \in X_s$.

Proof. Let iv) be satisfied and let $t \geq s \geq 0$ and $x \in X_s$. Then there are $n \in \mathbb{N}$ and $r \in [0, 1)$ such that $t - s = n + r$. We have

$$\|\Phi(t, s)x\| \leq p(t - s - n)\|\Phi(s + n, s)x\| \leq p(1)^{n+1}\|x\|.$$

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Let $\omega > \max\{0, \ln p(1)\}$. Then

$$\|\Phi(t,s)x\| \le p(1)e^{\omega n}\|x\| \le p(1)e^{\omega(t-s)}\|x\|.$$

The converse is obviously. □

In the sequel we will denote by M and ω those constants which satisfy condition v).

Definition 2. The evolution operator $\Phi(\cdot,\cdot)$ will be called exponentially stable, if there are $\nu > 0$ and a function $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \le N(t_0)e^{-\nu(t-t_0)}\|x\|,$$

for all $t \ge t_0 \ge 0$ and $x \in X_{t_0}$.

Remark 2. Let $\Phi(\cdot,\cdot)$ be an evolution operator. The following assertions are equivalent:

(1) $\Phi(\cdot,\cdot)$ is exponentially stable.

(2) There are $\nu > 0$ and $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \le N(s)e^{-\nu(t-s)}\|\Phi(s,t_0)x\|,$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X_{t_0}$.

Definition 3. The evolution operator $\Phi(\cdot,\cdot)$ will be called uniformly exponentially stable, if there are $N, \nu > 0$ such that

$$\|\Phi(t,t_0)x\| \le Ne^{-\nu(t-t_0)}\|x\|,$$

for all $t \ge t_0 \ge 0$ and $x \in X_{t_0}$.

Remark 3. The evolution operator $\Phi(\cdot,\cdot)$ is uniformly exponentially stable if and only if there are $N, \nu > 0$ such that

$$\|\Phi(t,t_0)x\| \le Ne^{-\nu(t-s)}\|\Phi(s,t_0)x\|,$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X_{t_0}$.

Lemma. *Let $\Phi(\cdot,\cdot)$ be an evolution operator. If there are $r > 0$ and a continuous function $g : [r,\infty) \to (0,\infty)$ such that*

$$\begin{cases} \inf\limits_{t>r} g(t) < 1, \\ \|\Phi(t,t_0)x\| \le g(t-t_0)\|x\|, \text{ for all } t_0 \ge 0,\ t \ge t_0 + r \text{ and } x \in X_{t_0}, \end{cases}$$

then $\Phi(\cdot,\cdot)$ is uniformly exponentially stable.

Proof. Let $\delta > r$ such that $g(\delta) < 1$.

For $t \geq t_0 \geq 0$ there is $n \in \mathbb{N}$ such that $n\delta \leq t - t_0 < (n+1)\delta$.

Let $x \in X_{t_0}$. Then

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-n\delta-t_0)}\|\Phi(t_0+n\delta,t_0)x\| \leq$$

$$\leq Me^{\omega(t-n\delta-t_0)}g(\delta)^n\|x\|.$$

Denoting $\nu = \dfrac{-\ln g(\delta)}{\delta} > 0$, it follows that

$$\|\Phi(t,t_0)x\| \leq Me^{\omega\delta}e^{\nu\delta}e^{-\nu(t-t_0)}\|x\|.$$

Denoting $N = Me^{(\omega+\nu)\delta}$, we obtain

$$\|\Phi(t,t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|,$$

for $t \geq t_0 \geq 0$ and $x \in X_{t_0}$. $\square$

Theorem 1. *The evolution operator $\Phi(\cdot,\cdot)$ is uniformly exponentially stable if and only if there is $K \in (0,\infty)$ such that*

$$\int_t^\infty \left(\int_u^{u+1} \|\Phi(s,t)x\|ds\right) du \leq K\|x\|, \text{ for all } t \geq 0 \text{ and } x \in X_t.$$

Proof. Let $\Phi(\cdot,\cdot)$ be an evolution operator which satisfy, for a $K > 0$, the condition of the hypothesis. We have

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-s)}\|\Phi(s,t_0)x\|,$$

so

$$e^{\omega s}\|\Phi(t,t_0)x\| \leq Me^{\omega t}\|\Phi(s,t_0)x\|, \text{ for } t \geq s \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

Let $t \geq t_0 + 1$. Integrating successively the last relation we obtain

$$\frac{1}{\omega}(e^\omega - 1)e^{\omega u}\|\Phi(t,t_0)x\| \leq Me^{\omega t}\int_u^{u+1} \|\Phi(s,t_0)x\|ds,$$

for $u \in [t_0, t-1]$, and so

$$\frac{e^\omega - 1}{\omega^2}(e^{\omega t-\omega} - e^{\omega t_0})\|\Phi(t,t_0)x\| \leq$$

$$\leq Me^{\omega t}\int_{t_0}^{t-1}\left(\int_u^{u+1} \|\Phi(s,t_0)x\|ds\right) du \leq MKe^{\omega t}\|x\|.$$

It follows that

$$e^{-\omega}\|\Phi(t,t_0)x\| \leq e^{-\omega(t-t_0)}\|\Phi(t,t_0)x\| + \frac{MK\omega^2}{e^{\omega}-1}\|x\| \leq$$

$$\leq M\left(1+\frac{K\omega^2}{e^{\omega}-1}\right)\|x\|.$$

For $t_0 \leq t < t_0+1$ and $x \in X_{t_0}$ we have

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-t_0)}\|x\| \leq Me^{\omega}\|x\|.$$

Denoting $L = Me^{\omega}\left(1+\dfrac{K\omega^2}{e^{\omega}-1}\right)$. we obtain

$$\|\Phi(t,t_0)x\| \leq L\|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

It follows that, for $t \geq s \geq t_0 \geq 0$, $x \in X_{t_0}$, we have

$$\|\Phi(t,t_0)x\| = \|\Phi(t,s)\Phi(s,t_0)x\| \leq L\|\Phi(s,t_0)x\|.$$

When $t \geq t_0+1$, we obtain

$$\|\Phi(t,t_0)x\| \leq L\int_u^{u+1}\|\Phi(s,t_0)x\|ds, \text{ for all } u \in [t_0, t-1],$$

and so

$$(t-1-t_0)\|\Phi(t,t_0)x\| \leq L\int_{t_0}^{t-1}\left(\int_u^{u+1}\|\Phi(s,t_0)x\|ds\right)du \leq LK\|x\|.$$

It follows by the preceding lemma that $\Phi(\cdot,\cdot)$ is uniformly exponentially stable.

The converse is immediately by direct calculation. □

Theorem 2. *Let $\Phi(\cdot,\cdot)$ be an evolution operator. If there are $\alpha > 0$ and a function $H(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that*

$$\int_t^{\infty}\left(\int_u^{u+1}e^{\alpha(s-t)}\|\Phi(s,t)x\|ds\right)du \leq H(t)\|x\|, \text{ for all } t \geq 0 \text{ and } x \in X_t,$$

then there is a function $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ such that

$$\|\Phi(t,t_0)x\| \leq N(t_0)e^{-\alpha(t-t_0)}\|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X_{t_0}.$$

Hence $\Phi(\cdot,\cdot)$ will be exponentially stable.

Proof. Let $t_0 \geq 0$, $t \geq t_0+1$ and $x \in X_{t_0}$. We have

$$\|\Phi(t,t_0)x\| \leq Me^{\omega(t-s)}\|\Phi(s,t_0)x\|, \text{ for } s \in [t_0,t].$$

It follows that

$$e^{-\alpha t_0} e^{(\omega+\alpha)s} \|\Phi(t,t_0)x\| \le M e^{\omega t} e^{\alpha(s-t_0)} \|\Phi(s,t_0)x\|,$$

and by integration, for $u \in [t_0, t-1]$, we have

$$e^{-\alpha t_0} \frac{e^{\omega+\alpha}-1}{\omega+\alpha} e^{(\omega+\alpha)u} \|\Phi(t,t_0)x\| \le M e^{\omega t} \int_u^{u+1} e^{\alpha(s-t_0)} \|\Phi(s,t_0)x\| ds,$$

and so

$$(e^{(\omega+\alpha)(t-1)} - e^{(\omega+\alpha)t_0}) \|\Phi(t,t_0)x\| \le$$
$$\le M e^{\omega t} \int_{t_0}^{t-1} \left(\int_u^{u+1} e^{\alpha(s-t_0)} \|\Phi(s,t_0)x\| ds \right) du,$$

from which

$$(e^{\alpha(t-t_0)-(\omega+\alpha)} - e^{-\omega(t-t_0)}) \|\Phi(t,t_0)x\| \le M \frac{(\omega+\alpha)^2}{e^{\omega+\alpha}-1} H(t_0) \|x\|.$$

It follows that

$$e^{\alpha(t-t_0)} \|\Phi(t,t_0)x\| \le e^{\omega+\alpha} \left(M \frac{(\omega+\alpha)^2}{e^{\omega+\alpha}-1} H(t_0) + M \right) \|x\|.$$

Denoting $N(t_0) = M e^{\omega+\alpha} \left(\frac{(\omega+\alpha)^2}{e^{\omega+\alpha-1}} H(t_0) + 1 \right)$, we obtain

$$\|\Phi(t,t_0)x\| \le N(t_0) e^{-\alpha(t-t_0)} \|x\|.$$

For $t_0 \le t < t_0 + 1$ and $x \in X_{t_0}$ we have

$$\|\Phi(t,t_0)x\| \le M e^{\omega} e^{\alpha} e^{-\alpha(t-t_0)} \|x\|.$$

So, it follows that

$$\|\Phi(t,t_0)x\| \le N(t_0) e^{-\alpha(t-t_0)} \|x\|, \text{ for all } t \ge t_0 \ge 0 \text{ and } x \in X_{t_0}. \ \square$$

Using in the proofs of the theorems the p power of the norm, respectively the p power of the inner integral ($p \in [1,\infty)$), we obtain the following results.

Corollary 1. *Let $\Phi(\cdot,\cdot)$ be an evolution operator and $p \in [1,\infty)$ be arbitrarily. The following assertions are equivalent.*

1) $\Phi(\cdot,\cdot)$ is uniformly exponentially stable.

2) There is $K \in (0,\infty)$ such that

$$\int_t^\infty \left(\int_u^{u+1} \|\Phi(s,t)x\|^p ds \right) du \le K \|x\|^p, \text{ for all } t \ge 0 \text{ and } x \in X_t.$$

3) There is $K \in (0,\infty)$ *such that*

$$\int_t^\infty \left(\int_u^{u+1} \|\Phi(s,t)x\| ds\right)^p du \le K\|x\|^p, \text{ for all } t \ge 0 \text{ and } x \in X_t.$$

Corollary 2. *Let* $\Phi(\cdot,\cdot)$ *be an evolution operator and* $p \in [1,\infty)$ *be arbitrarily.*

1) If there are $\alpha > 0$ *and a function* $H(\cdot) : \mathbb{R}_+ \to (0,\infty)$ *such that*

$$\int_t^\infty \left(\int_u^{u+1} e^{\alpha(s-t)} \|\Phi(s,t)x\|^p ds\right) du \le H(t)\|x\|^p, \text{ for all } t \ge 0 \text{ and } x \in X_t,$$

then there is $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ *such that*

$$\|\Phi(t,t_0)x\| \le N(t_0) e^{-\frac{\alpha}{p}(t-t_0)} \|x\|, \text{ for all } t \ge t_0 \ge 0 \text{ and } x \in X_{t_0}.$$

2) If there are a function $H(\cdot) : \mathbb{R}_+ \to (0,\infty)$ *and* $\alpha > 0$ *such that*

$$\int_t^\infty \left(\int_u^{u+1} e^{\alpha(s-t)} \|\Phi(s,t)x\| ds\right)^p du \le H(t)\|x\|^p, \text{ for all } t \ge 0 \text{ and } x \in X_t,$$

then there is $N(\cdot) : \mathbb{R}_+ \to (0,\infty)$ *such that*

$$\|\Phi(t,t_0)x\| \le N(t_0) e^{-\alpha(t-t_0)} \|x\|, \text{ for all } t \ge t_0 \ge 0 \text{ and } x \in X_{t_0}.$$

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