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ON THE OSTROWSKI'S INTEGRAL INEQUALITY FOR LIPSCHITZIAN MAPPINGS AND APPLICATIONS

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Abstract. A generalization of Ostrowski's inequality for lipschitzian mappings and applications in Numerical Analysis and for Euler's Beta function are given.

1. INTRODUCTION

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [2, p. 469].

THEOREM 1.1. *Let $f : [a, b] \to R$ be a differentiable mapping on (a, b) whose derivative is bounded on (a, b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a, b]$ we have the inequality*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty.$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In this paper we prove that Ostrowski's inequality also holds for lipschitzian mappings and apply it in obtaining a Riemann's type quadrature formula for this class of mappings. Applications for Euler's Beta function are also given.

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2. OSTROWSKI'S INEQUALITY FOR LIPSCHITZIAN MAPPINGS

The following inequality for lipschitzian mappings holds:

THEOREM 2.1. *Let $u : [a,b] \to R$ be an $L-$lipschitzian mapping on $[a,b]$, i.e.,*

$$|u(x) - u(y)| \leq L|x - y| \text{ for all } x, y \in [a,b].$$

Then we have the inequality

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq L(b-a)^2 \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right]. \tag{2.1}$$

for all $x \in [a,b]$.

The constant $\frac{1}{4}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral we have

$$\int_a^x (t-a)du(t) = u(x)(x-a) - \int_a^x u(t)dt$$

and

$$\int_x^b (t-b)du(t) = u(x)(b-x) - \int_x^b u(t)dt.$$

If we add the above two equalities, then we get

$$u(x)(b-a) - \int_a^b u(t)dt = \int_a^b p(x,t)du(t) \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a \text{ if } t \in [a,x) \\ t-b \text{ if } x \in [x,b] \end{cases},$$

for all $x, t \in [a,b]$.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$. If $p : [a,b] \to R$ is Riemann integrable on $[a,b]$ and $v : [a,b] \to R$ is L-lipschitzian on $[a,b]$, then

$$\left| \int_a^b p(x)dv(x) \right| = \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)})[v(x_{i+1}^{(n)}) - v(x_i^{(n)})] \right|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} \left| p(\xi_i^{(n)}) \right| (x_{i+1}^{(n)} - x_i^{(n)}) \left| \frac{v(x_{i+1}^{(n)}) - v(x_i^{(n)})}{x_{i+1}^{(n)} - x_i^{(n)}} \right|$$

$$\leq L \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} |p(\xi_i^{(n)})|(x_{i+1}^{(n)} - x_i^{(n)}) = L \int_a^b |p(x)|dx. \tag{2.3}$$

Applying the inequality (2.3) for $p(x,t)$ as above and $v(x) = u(x), x \in [a,b]$, we get

$$\left| \int_a^b p(x,t)du(t) \right| \leq L \int_a^b |p(x,t)|dt$$

$$= L[\int_a^x \left[t - a|dt + \int_x^b |t-b|dt \right] = \frac{L}{2}[(x-a)^2 + (b-x)^2]$$

$$= L(b-a)^2[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}] \tag{2.4}$$

and then by (2.4), via the identity (2.2), we deduce the desired inequality (2.1).

Now, assume that the inequality (2.1) holds with a constant $C > 0$, i.e.,

$$\left| \int_a^b u(t)dt - u(x)(b-a) \right| \leq L(b-a)^2 \left[C + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \tag{2.5}$$

for all $x \in [a,b]$.

Consider the mapping $f : [a,b] \to R, f(x) = x$ in (2.5). Then

$$\left| x - \frac{a+b}{2} \right| \leq C + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}$$

for all $x \in [a,b]$; and then for $x = a$, we get

$$\frac{b-a}{2} \leq \left(C + \frac{1}{4} \right)(b-a)$$

which implies that $C \geq \frac{1}{4}$ and the theorem is completely proved. □

The following corollary holds:

COROLLARY 2.2. *Let $u : [a,b] \to R$ be as above. Then we have the inequality:*

$$\left| \int_a^b u(t)dx - u(\frac{a+b}{2})(b-a) \right| \leq \frac{1}{4}L(b-a)^2. \tag{2.6}$$

Remark 2.3. It is well known that if $f : [a, b] \to R$ is a convex mapping on $[a, b]$, then *Hermite-Hadamard's* inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2.7}$$

Now, if we assume that $f : I \subset R \to R$ is convex on I and $a, b \in Int(I), a < b$, then f'_+ is monotonous nondecreasing on $[a, b]$ and by Theorem 2.1 we get

$$0 \leq \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \leq \frac{1}{4}f'_+(b)(b-a) \tag{2.8}$$

which gives a counterpart for the first membership of Hadamard's inequality.

3. A QUADRATURE FORMULA OF RIEMANN TYPE

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \xi) = \sum_{i=0}^{n-1} f(\xi_i)h_i$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula

THEOREM 3.1. *Let $f : [a, b] \to R$ be an $L-$lipschitzian mapping on $[a, b]$ and I_n, ξ_i $(i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \xi) + W_n(f, I_n, \xi) \tag{3.1}$$

where the remainder satisfies the estimation

$$|W_n(f, I_n, \xi)| \leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2 + L\sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2$$
$$\leq \frac{1}{2}L\sum_{i=0}^{n-1} h_i^2 \tag{3.2}$$

for all ξ_i $(i = 0, ..., n-1)$ as above.

The constant $\frac{1}{4}$ is sharp in (3.2).

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right| \leq L\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]. \tag{3.3}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$|W_n(f, I_n, \xi)| \leq \sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i\right|$$

$$\leq L\sum_{i=0}^{n-1}\left[\frac{1}{4}h_i^2 + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right].$$

Now, as

$$\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2 \leq \frac{1}{4}h_i^2$$

for all $\xi_i \in [x_i, x_{i+1}](i = 0, ..., n-1)$ the second part of (3.2) is also proved. □

Note that the best estimation we can get from (3.2) is that one for which $\xi_i = \frac{x_i+x_{i+1}}{2}$ obtaining the following midpoint formula:

COROLLARY 3.2. *Let f, I_n be as above. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$|S_n(f, I_n)| \leq \frac{1}{4}L\sum_{i=0}^{n-1} h_i^2.$$

Remark 3.3. If we assume that $f : [a, b] \to R$ is differentiable on (a, b) and whose derivative f' is bounded on (a, b) we can put instead of L the infinity norm $\|f'\|_\infty$ obtaining the estimation due to Dragomir-Wang from the paper [1].

4. APPLICATIONS FOR EULER'S BETA MAPPING

Consider the mapping Beta for real numbers

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \; p,q>0$$

and the mapping $e_{p,q}(t) := t^{p-1}(1-t)^{q-1}, t \in [0,1]$.

We have for $p,q>1$ that

$$e'_{p,q}(t) = e_{p-1,q-1}(t)[p-1-(p+q-2)t].$$

If $t \in \left[0, \frac{p-1}{p+q-2}\right)$ then $e'_{p,q}(t) > 0$ and if $t \in \left(\frac{p-1}{p+q-2}, 1\right]$ then $e'_{p,q}(t) < 0$ which shows that for $t_0 = \frac{p-1}{p+q-2}$ we have a maximum for $e_{p,q}$ and then:

$$\sup_{t\in[0,1]} e_{p,q}(t) = e_{p,q}(t_0) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{(p+q-2)^{p+q-2}}; \; p,q>1.$$

Consequently

$$|e'_{p,q}(t)| \le \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \max_{t\in[0,1]} |p-1-(p+q-2)t|$$
$$= \max\{p-1,q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}; \; p,q>2$$

for all $t \in [0,1]$ and then

$$\| e'_{p,q}\|_\infty \le \max\{p-1,q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \; p,q>2. \tag{4.1}$$

The following inequality for Beta mapping holds

PROPOSITION 4.1. *Let $p,q>2$ and $x \in [0,1]$. Then we have the inequality*

$$|B(p,q) - x^{p-1}(1-x)^{q-1}|$$
$$\le \max\{p-1,q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} [\frac{1}{4} + (x-\frac{1}{2})^2]. \tag{4.2}$$

The proof follows by Theorem 2.1 applied for the mapping $e_{p,q}$ and taking into account that $\| e'_{p,q}\|_\infty$ satisfies the inequality (4.1).

COROLLARY 4.2. *Let $p, q > 2$. Then we have the inequality*

$$\left| B(p,q) - \frac{1}{2^{p+q-2}} \right| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}.$$

Now, if we apply Theorem 3.1 for the mapping $e_{p,q}$ we get the following approximation of Beta mapping in terms of Riemann sums.

PROPOSITION 4.3. *Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n and $p, q > 2$. Then we have the formula*

$$B(p,q) = \sum_{i=0}^{n-1} \xi_i^{p-1} (1-\xi_i)^{q-1} h_i + T_n(p,q)$$

where the remainder $T_n(p,q)$ satisfies the estimation

$$|T_n(p,q)| \leq \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}}$$

$$\times \left[\frac{1}{4} \sum_{i=0}^{n-1} h_i^2 + \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2} \right)^2 \right]$$

$$\leq \frac{1}{2} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

Particularly, if we choose above $\xi_i = \frac{x_i + x_{i+1}}{2}$ $(i = 0, ..., n-1)$ then we get the approximation

$$B(p,q) = \frac{1}{2^{p+q-2}} \sum_{i=0}^{n-1} (x_i + x_{i+1})^{p-1} (2 - x_i - x_{i+1})^{q-1} + V_n(p,q)$$

where

$$|V_n(p,q)| \leq \frac{1}{4} \max\{p-1, q-1\} \frac{(p-2)^{p-2}(q-2)^{q-2}}{(p+q-4)^{p+q-4}} \sum_{i=0}^{n-1} h_i^2.$$

References

[1] Dragomir, S.S. and Wang, S., Applications of Ostrowski's inequality to the estimation of error bounds for some special means and for some numerical quadrature rules, *Appl. Math. Lett.*, **11**(1)(1998), 105-109.
[2] Mitrinović, D.S., Pečarić, J.E. and Fink, A.M., *Inequalities for Functions and their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.

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