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SOME PROPERTIES OF THE INTEGRAL OPERATORS IN UNIVALENT FUNCTION

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Abstract. In this paper we have obtained some properties of the integral operators on the lines of Miller and Mocanu [2], Nour [4], after generalizing several lemmas of the above mentioned authors needed in the course of research.

1. **Introduction**

Let $\mathcal{A}$ denote the class of functions analytic in the unit disc $U = \{z : \ |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Also let S denote the subclass of $\mathcal{A}$ consisting of (normalized) functions f which are univalent in U. A function $f(z)$ in S is said to be starlike of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in U,\ 0 \leq \alpha < 1).$$

Let $S^*(\alpha)$ denote the class of all functions which are starlike of order α in U. It is well known that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$.

Let f, g be analytic in the unit disc U. We call the function f is a subordinate to g, written $f \prec g$, if there exists an analytic function ϕ with $\phi(0) = 0$ and $|\phi(z)| < 1$ such that $f(z) = g(\phi(z))$.

Let $\rho(A, B)$ consist of all functions g that are analytic in U with $g(0) = 1$ and satisfy the condition

$$g(z) \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1).$$

Finally a function $f(z) \in \mathcal{A}$ is said to be in the class $S^*(A, B)$ if and only if

$$\frac{zf'(z)}{f(z)} \in \rho(A, B).$$

In the present paper we will investigate some properties of the integral operators. We shall make use of the results due to Miller and Mocanu [2] and Noor [4]. For the sake of convenience, we recall those results as the following lemmas:

Lemma 1 (Miller and Mocanu [2]). *Let $\alpha \geq 0$, $\beta > 0$ and $\alpha + \delta = \beta + \gamma > 0$ and let the functiona $\varphi(z)$ and $\phi(z)$ be in the class D defined by*

$$D := \{\theta : \ \theta(z) \text{ analytic in } U, \ \theta(z) \neq 0, \ \text{and } \theta(0) = 1\}.$$

Suppose also that

$$\delta + Re\left\{\frac{z\varphi'(z)}{\varphi(z)}\right\} \geq \gamma \quad and \quad Re\left\{\frac{z\phi'(z)}{\phi(z)}\right\} \leq \beta w(0)$$

where $w(\rho)$ is given, in terms of the Gaussian hypergeometric function ${}_2F_1$, by

$$w(\rho) = \frac{1}{\beta}\left[\frac{(\beta+\gamma)2^{-2\beta(1-\rho)}}{{}_2F_1[2\beta(1-\rho), \beta+\gamma; \beta+\gamma+1; -1]} - \gamma\right]$$
$$(\max\{(\beta-\gamma-1)/2\beta, -\gamma/\beta\} \leq \rho < 1)$$

Then for the integral operator I defined by

$$I(f)(z) = \left(\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z \{f(t)\}^{\alpha}\varphi(t)t^{\delta-1}dt\right)^{1/\beta}$$

we have

$$I(S^*) \subset \begin{cases} S^* & (\phi(z) \not\equiv 1) \\ S^*(w(0)) & (\phi(z) \equiv 1) \end{cases}$$

Lemma 2 (Noor [4]). *Let $\rho_j(z) \in \rho(A, B)$, $(j = 1, 2)$. Then, for $\alpha > 0$ and $\beta > 0$,*

$$\frac{\alpha\rho_1(z) + \beta\rho_2(z)}{\alpha+\beta} \in \rho(A, B).$$

2. Some results related to the function space $\rho(A, B)$

Lemma 3. *Let $\alpha \geq 0$ and $D(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the region. Let $N(z)$ be analytic in E with $N(0) = D(0) = 0$.*

Then

$$(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)} \prec \frac{1+Az}{1+Bz} \ \Rightarrow \ \frac{N(z)}{D(z)} \prec \frac{1+Az}{1+Bz}$$

where $(1 \leq B < A \leq 1)$.

Proof. Let

$$\frac{N(z)}{D(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}.$$

Clearly $w(0) = 0$. We will prove that $|w(z)| < 1$, $\forall\ z \in U$ for, if not, by Jack's lemma [1] there exists $z_0 \in U$, such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. We consider

$$\varphi(z) = (1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)}$$

since

$$\frac{N'(z)}{D'(z)} = \frac{N(z)}{D(z)} + \frac{D(z)}{D'(z)}\left(\frac{(A-B)w'(z)}{(1+Bw(z))^2}\right).$$

So

$$\varphi(z_0) = (1-\alpha)\frac{N(z_0)}{D(z_0)} + \alpha\frac{N'(z_0)}{D'(z_0)} =$$

$$= \frac{N(z_0)}{D(z_0)} + \alpha\left(\frac{D(z_0)}{z_0 D'(z_0)}\right)\left(\frac{(A-B)kw(z_0)}{(1+Bw(z_0))^2}\right).$$

Now

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| = \left|\frac{\dfrac{(A-B)w(z_0)}{1+Bw(z_0)}\left(1+\dfrac{D(z_0)}{z_0D'(z_0)}\dfrac{k\alpha}{1+Bw(z_0)}\right)}{\dfrac{(B-A)}{1+Bw(z_0)}\left(1-\dfrac{D(z_0)k\alpha Bw(z_0)}{z_0D'(z_0)(1+Bw(z_0))}\right)}\right|$$

or

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| = \left|\frac{1+\dfrac{D(z_0)k\alpha}{z_0D'(z_0)(1+Bw(z_0))}}{1-\dfrac{D(z_0)k\alpha\beta w(z_0)}{z_0D'(z_0)(1+\beta w(z_0))}}\right|$$

Therefore

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| > 1 \ \Leftrightarrow\ \left|1+\frac{k\alpha D(z_0)}{z_0D'(z_0)(1+Bw(z_0))}\right| >$$

$$> \left|1-\frac{k\alpha w(z_0)D(z_0)}{z_0D'(z_0)(1+Bw(z_0))}\right|$$

But

$$\left|1+\frac{k\alpha D(z_0)}{z_0D'(z_0)(1+Bw(z_0))}\right|^2 - \left|1-\frac{k\alpha Bw(z_0)D(z_0)}{z_0D'(z_0)(1+Bw(z_0))}\right|^2 =$$

$$= (1-B)^2\left|\frac{D(z_0)}{z_0D'(z_0)}\right|^2\left|\frac{k\alpha}{1+Bw(z_0)}\right|^2 + 2k\alpha Re\left(\frac{D(z_0)}{z_0D'(z_0)}\right) > 0.$$

Hence

$$\left|\frac{\varphi(z_0)-1}{B\varphi(z_0)-A}\right| > 1$$

and this is contradiction with this fact that $\varphi(z) \prec \dfrac{1+Az}{1+Bz}$ so $|w(z)| < 1$ and the proof is complete.

By putting $\alpha = 0$ we get the result due to Miller and Mocanu [3] as:

Corollary 1. *Let the functions $M(z)$ and $N(z)$ be analytic in U with $M(0) = N(0) = 0$ and let γ be a real number. Suppose also that $N(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the origin. Then*

$$Re\left\{\frac{M'(z)}{N'(z)}\right\} > \gamma, \quad (z \in U) \ \Rightarrow \ Re\left(\frac{M(z)}{N(z)}\right) > \gamma, \ (z \in U).$$

Lemma 4. *Let $\alpha \geq 0$ and $D(z)$ maps U onto a (possibly many-sheeted) region which is starlike with respect to the region. Let $N(z)$ be analytic in E with $N(0) = D(0) = 0$ and $\dfrac{N'(0)}{D'(0)} = k$ then*

$$(1-\alpha)\frac{N(z)}{kD(z)} + \alpha\frac{N'(z)}{kD'(z)} \prec \frac{1+Az}{1+Bz} \ \Rightarrow \ \frac{N(z)}{kD(z)} \prec \frac{1+Az}{1+Bz}$$

(where $-1 \leq B < A \leq 1$).

Proof. Proceeding as in the proof of Lemma 3 we get our result.

By putting $\alpha = 0$ we get the result due to Reddy and Padmanabhan [5] as:

Corollary 2. *Let the functions $N(z)$ and $D(z)$ be analytic in U and let $D(z)$ maps U onto a many-sheeted starlike region. Suppose also that $N(0) = D(0) = 0$, $\dfrac{N'(0)}{D'(0)} = k$ and $\dfrac{N'(z)}{kD'(z)} \in \rho(A,B)$, $(k \geq 1)$ then $\dfrac{N(z)}{kD(z)} \in \rho(A,B)$.*

Lemma 5. *Let $\alpha > 0$ and $f \in \mathcal{A}$. Then*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) + \lambda\left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A,B) \ \Rightarrow \ \left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A,B)$$

(where $-1 \leq B < A \leq 1$ and $0 \leq \lambda \leq 1$).

Proof. Let

$$\left(\frac{f(z)}{z}\right)^{\alpha} = \frac{1+Aw(z)}{1+Bw(z)}.$$

Clearly $w(0) = 0$. We will prove $|w(z)| < 1$, $\forall\ z \in U$. For, if not, by Jack's lemma [1] there exists $z_0 \in E$, such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. Let

$$\psi(z) = (1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) + \lambda\left(\frac{f(z)}{z}\right)^{\alpha}.$$

But

$$\alpha\left(\frac{zf'(z)-f(z)}{z^2}\right)\left(\frac{f(z)}{z}\right)^{\alpha-1}=\frac{(A-B)w'(z)}{(1+Bw(z))^2}$$

or

$$f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}=\frac{1+Aw(z)}{1+Bw(z)}+\frac{(A-B)zw'(z)}{\alpha(1+Bw(z))^2}$$

Hence

$$\psi(z_0)=\frac{1+Aw(z_0)}{1+Bw(z_0)}+\frac{(1-\lambda)kw(z_0)(A-B)}{\alpha(1+Bw(z_0))^2}$$

If we take $\phi(z)=\dfrac{(1-\lambda)k}{\alpha(1+Bw(z))}$ then we have

$$\left|\frac{\psi(z_0)-1}{B\psi(z_0)-A}\right|=\left|\frac{\frac{(A-B)w(z_0)}{1+Bw(z_0)}\left(1+\frac{(1-\lambda)k}{\alpha(1+Bw(z_0))}\right)}{\frac{B-A}{1+Bw(z_0)}\left(1-\frac{(1-\lambda)kw(z_0)B}{\alpha(1+Bw(z_0))}\right)}\right|=$$

$$=\left|\frac{1+\phi(z_0)}{1-\phi(z_0)Bw(z_0)}\right|$$

But the right hand side of above equivality is greater than 1, because

$$|1+\phi(z_0)|^2-|1-Bw(z_0)\phi(z_0)|^2=(1-B^2)|\phi(z_0)|^2+\frac{2(1-\lambda)k}{\alpha}>0$$

and this is contradiction with hypothesis, so $|w(z)|<1$ and the proof is complete. By putting $\lambda=0$ we get the result due to Noor [4] as

Corollary 3. *If $f(z)\in\mathcal{A}$ and $\left(\frac{f(z)}{z}\right)^{\alpha-1}f'(z)\in\rho(A,B)$ then $\left(\frac{f(z)}{z}\right)^{\alpha}\in\rho(A,B)$ (where $\alpha\in N=\{1,2,3,\dots\}$).*

3. Some properties of the integral operators

Theorem 1. *Let $g\in S^*(A,B)$, then the function $F(z)$ defined by*

$$F(z)=\left[\alpha^{-1}\int_0^z g(t)^{1/\alpha}t^{-1}dt\right]^{\alpha}$$

is in the class $S^*(A,B)$, $(\alpha>0)$.

Proof. We know from Lemma 1 that $F(z)\in S^*$. But with direct calculation we can write

$$\frac{zg'(z)}{g(z)}=(1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)$$

So, by hypothesis,

$$(1-\alpha)\frac{zF'(z)}{F(z)}+\alpha\left(1+\frac{zF''(z)}{F'(z)}\right)\in\rho(A,B). \tag{3.1}$$

We consider $N(z) = zF'(z)$ and $D(z) = F(z)$, then functions $N(z)$ and $D(z)$ satisfy the conditions of Lemma 3. Now from (3.1) we have

$$(1-\alpha)\frac{N(z)}{D(z)} + \alpha\frac{N'(z)}{D'(z)} \in \rho(A, B).$$

So, by lemma 3,

$$\frac{zF'(z)}{F(z)} = \frac{N(z)}{D(z)} \in \rho(A, B)$$

and this completes the proof.

Theorem 2. *Let $\alpha > 0$, $\gamma > 0$, $f(z) \in \mathcal{A}$ and $F(z)$ be defined by*

$$F(z) = \left(\frac{\alpha+\gamma}{z^{\gamma}} \int_0^z f(t)^{\alpha} t^{\gamma-1} dt\right)^{1/\alpha}$$

then

$$\left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A, B) \;\Rightarrow\; \left(\frac{F(z)}{z}\right)^{\alpha} \in \rho(A, B).$$

Proof. Since

$$\alpha F'(z) = \left(\frac{-\gamma(\alpha+\gamma)}{z^{\gamma+1}} \int_0^z f(t)^{\alpha} t^{\gamma-1} dt + \frac{\alpha+\gamma}{z^{\gamma}} f(z)^{\alpha} z^{\gamma-1}\right) F(z)^{1-\alpha} =$$

$$= \left(-\frac{\gamma}{z} F(z)^{\alpha} + \frac{\alpha+\gamma}{z} f(z)^{\alpha}\right) F(z)^{1-\alpha}$$

or

$$\frac{\alpha}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha-1} + \frac{\gamma}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha} = \left(\frac{f(z)}{z}\right)^{\alpha} \tag{3.2}$$

But, by hypothesis, $\left(\frac{f(z)}{z}\right)^{\alpha} \in \rho(A, B)$. Therefore from (3.2) we have

$$\frac{\alpha}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha-1} F'(z) + \frac{\gamma}{\alpha+\gamma}\left(\frac{F(z)}{z}\right)^{\alpha} \in \rho(A, B) \tag{3.3}$$

Hence from (3.3) and Lemma 5 we get the desired result.

Theorem 3. *Let $\alpha > 1$, $f, g \in \mathcal{A}$ and function $F(z)$ is defined by*

$$F(z) = \left[\alpha^{-1} \int_0^z f(t)^{1/\alpha} g(t)^{(\alpha-1)/\alpha_t - 1} dt\right]^{\alpha}. \tag{3.4}$$

Then $\frac{zg'(z)}{g(z)} \in \rho(A, B)$ *and* $\frac{zf'(z)}{f(z)} \in \rho(A, B) \;\Rightarrow\; \frac{1}{\alpha}\frac{zF'(z)}{F(z)} \in \rho(A, B)$.

Proof. It is clear, by Lemma 1, $F \in S^*$. By differentiation from (3.4) we get

$$F'(z) = (f(z)^{1/\alpha} g(z)^{(\alpha-1)/\alpha_z - 1})(F(z))^{(\alpha-1)/\alpha}$$

or

$$zF(z)^{(1-\alpha)/\alpha}F'(z) = f(z)^{1/\alpha}g(z)^{(\alpha-1)/\alpha}. \tag{3.5}$$

By differentiation from (3.5) we get

$$\left(1+\frac{zF''(z)}{F'(z)}\right)+\left(\frac{1-\alpha}{\alpha}\right)\frac{zF'(z)}{F(z)}=\frac{1}{\alpha}\frac{zf'(z)}{f(z)}+\frac{\alpha-1}{\alpha}\frac{zg'(z)}{g(z)}.$$

But the right habd side of the above equality belongs to $\rho(A,B)$, by lemma 2. So we have

$$\left(1+\frac{zF''(z)}{F'(z)}\right)+\left(\frac{1-\alpha}{\alpha}\right)\frac{zF'(z)}{F(z)}\in\rho(A,B). \tag{3.6}$$

Let $N(z) = zF'(z)$ and $D(z) = \alpha F(z)$ then functions $N(z)$ and $D(z)$ satisfy the condition of Lemma 3. But

$$\left(1+\frac{zF''(z)}{F'(z)}\right)+\frac{1-\alpha}{\alpha}\frac{zF'(z)}{F(z)}=\alpha\frac{N'(z)}{D'(z)}+(1-\alpha)\frac{N(z)}{D(z)} \tag{3.7}$$

So from relations (3.6), (3.7) and lemma 3 we have $\dfrac{N(z)}{D(z)}=\dfrac{zF'(z)}{\alpha F(z)}\in\rho(A,B)$ and the proof is complete.

References

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