Shepard operator of least squares thin-plate spline type

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Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. We obtain some new Shepard type operators based on the classical, the modified Shepard methods and the least squares thin-plate spline function. Given some sets of points, we compute some representative subsets of knot points following an algorithm described by J. R. McMahon in 1986.

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1. Preliminaries

One of the best suited methods for approximating large sets of data is the Shepard method, introduced in 1968 in [16]. It has the advantages of a small storage requirement and an easy generalization to additional independent variables, but it suffers from no good reproduction quality, low accuracy and a high computational cost relative to some alternative methods [14], these being the reasons for finding new methods that improve it (see, e.g.,[1]-[8], [17], [18]). In this paper we obtain some new operators based on the classical, the modified Shepard methods and the least squares thin-plate spline.

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$, $i = 1, ..., N$ some distinct points. Denote by $r_i(x, y)$ the distances between a given point $(x, y) \in X$ and the points (x_i, y_i), $i = 1, ..., N$. The bivariate Shepard operator is defined by

$$(S_\mu f)(x, y) = \sum_{i=1}^{N} A_{i,\mu}(x, y) f(x_i, y_i), \qquad (1.1)$$

where

$$A_{i,\mu}(x,y) = \frac{\prod\limits_{\substack{j=1 \\ j \neq i}}^{N} r_j^{\mu}(x,y)}{\sum\limits_{k=1}^{N} \prod\limits_{\substack{j=1 \\ j \neq k}}^{N} r_j^{\mu}(x,y)}, \tag{1.2}$$

with the parameter $\mu > 0$.

It is known that the bivariate Shepard operator S_μ reproduces only the constants and that the function $S_\mu f$ has flat spots in the neighborhood of all data points.

Franke and Nielson introduced in [10] a method for improving the accuracy in reproducing a surface with the bivariate Shepard approximation. This method has been further improved in [9], [15], [14], and it is given by:

$$(Sf)(x,y) = \frac{\sum\limits_{i=1}^{N} W_i(x,y) f(x_i,y_i)}{\sum\limits_{i=1}^{N} W_i(x,y)}, \tag{1.3}$$

with

$$W_i(x,y) = \left[\frac{(R_w - r_i)_+}{R_w r_i} \right]^2, \tag{1.4}$$

where R_w is a radius of influence about the node (x_i, y_i) and it is varying with i. R_w is taken as the distance from node i to the jth closest node to (x_i, y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g. [14]). As it is mentioned in [11], this modified Shepard method is one of the most powerful software tools for the multivariate approximation of large scattered data sets.

2. The Shepard operators of least squares thin-plate spline type

Consider f a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$, $i = 1, ..., N$ some distinct points. We introduce the Shepard operator based on the least squares thin-plate spline in four ways.

Method 1. *We consider*

$$(S_1 f)(x,y) = \sum_{i=1}^{N} A_{i,\mu}(x,y) F_i(x,y), \tag{2.1}$$

where $A_{i,\mu}$, $i = 1, ..., N$, *are defined by (1.2), for a given parameter* $\mu > 0$ *and the least squares thin-plate splines are given by*

$$F_i(x,y) = \sum_{j=1}^{i} C_j d_j^2 \log(d_j) + ax + by + c, \quad i = 1, ..., N, \tag{2.2}$$

with $d_j = \sqrt{(x - x_j)^2 + (y - y_j)^2}$.

For the second way, we consider a smaller set of $k \in \mathbb{N}^*$ knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$ that will be representative for the original set. This set is obtained following the next steps (see, e.g., [12] and [13]):

Algorithm 2.1.   1. Generate k random knot points, with $k < N$;
   2. Assign to each point the closest knot point;
   3. If there exist knot points for which there is no point assigned, move the knot to the closest point;
   4. Compute the next set of knot points as the arithmetic mean of all corresponding points;
   5. Repeat steps 2-4 until the knot points do not change for two successive iterations.

Method 2. *For a given $k \in \mathbb{N}^*$, we consider the representative set of knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$. The Shepard operator of least squares thin-plate spline is given by*

$$(S_2 f)(x, y) = \sum_{i=1}^{k} A_{i,\mu}(x, y) F_i(x, y), \tag{2.3}$$

where $A_{i,\mu}$, $i = 1, ..., k$, are defined by

$$A_{i,\mu}(x, y) = \frac{\displaystyle\prod_{\substack{j=1 \\ j \neq i}}^{k} r_j^{\mu}(x, y)}{\displaystyle\sum_{p=1}^{k} \prod_{\substack{j=1 \\ j \neq p}}^{k} r_j^{\mu}(x, y)},$$

for a given parameter $\mu > 0$.

The least squares thin-plate spline are given by

$$F_i(x, y) = \sum_{j=1}^{i} C_j d_j^2 \log(d_j) + ax + by + c, \quad i = 1, ..., k, \tag{2.4}$$

with $d_j = \sqrt{(x - \hat{x}_j)^2 + (y - \hat{y}_j)^2}$.

For Methods 1 and 2, the coefficients C_j, a, b, c of F_i are found such that to minimize the expressions

$$E = \sum_{i=1}^{N'} [F_i(x_i, y_i) - f(x_i, y_i)]^2,$$

considering $N' = N$ for the first case and $N' = k$ for the second one. There are obtained systems of the following form (see, e.g., [12]):

$$
\begin{pmatrix}
0 & d_{12}^2 \log d_{12} & \cdots & d_{1N'}^2 \log d_{1N'} & x_1 & y_1 & 1 \\
d_{21}^2 \log d_{21} & 0 & \cdots & d_{2N'}^2 \log d_{2N'} & x_2 & y_2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d_{N'1}^2 \log d_{N'1} & d_{N'2}^2 \log d_{N'2} & \cdots & 0 & x_{N'} & y_{N'} & 1 \\
x_1 & x_2 & \cdots & x_{N'} & 0 & 0 & 0 \\
y_1 & y_2 & \cdots & y_{N'} & 0 & 0 & 0 \\
1 & 1 & \cdots & 1 & 0 & 0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
C_1 \\ C_2 \\ \vdots \\ C_{N'} \\ a \\ b \\ c
\end{pmatrix}
=
\begin{pmatrix}
f_1 \\ f_2 \\ \vdots \\ f_{N'} \\ 0 \\ 0 \\ 0
\end{pmatrix}
$$

with $d_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$, $f_i = f(x_i, y_i)$, $i, j = 1, ..., N'$.

Next we consider the improved form of the Shepard operator given in (1.3).

Method 3. *We consider Shepard operator of least squares thin-plate spline type of the following form:*

$$(S_3 f)(x,y) = \frac{\sum\limits_{i=1}^{N} W_i\,(x,y)\,F_i(x,y)}{\sum\limits_{i=1}^{N} W_i\,(x,y)}, \qquad (2.5)$$

with W_i given by (1.4), F_i given by (2.2), for $i = 1, ..., N$.

The coefficients C_j, a, b, c of F_i, $i = 1, ..., N$ are determined in order to minimize the expression

$$E = \sum\limits_{i=1}^{N} [F_i(x_i, y_i) - f(x_i, y_i)]^2.$$

Method 4. *For a given $k \in \mathbb{N}^*$, we consider the representative set of knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$, obtained applying the Algorithm 2.1. In this case, we introduce the Shepard operator of least squares thin-plate spline type by the following formula:*

$$(S_4 f)(x,y) = \frac{\sum\limits_{i=1}^{k} W_i\,(x,y)\,F_i(x,y)}{\sum\limits_{i=1}^{k} W_i\,(x,y)}, \qquad (2.6)$$

with W_i given by (1.4) and F_i given by (2.4), for $i = 1, ..., k$.

The coefficients C_j, a, b, c of F_i, $i = 1, ..., k$ are determined in order to minimize the expression

$$E = \sum\limits_{i=1}^{k} [F_i(x_i, y_i) - f(x_i, y_i)]^2.$$

3. Numerical examples

We consider the following test functions (see, e.g., [9], [15], [14]):

Gentle: $f_1(x,y) = \exp[-\frac{81}{16}((x-0.5)^2 + (y-0.5)^2)]/3,$

Saddle: $f_2(x,y) = \dfrac{(1.25 + \cos 5.4y)}{6 + 6(3x-1)^2},$ (3.1)

Sphere: $f_3(x,y) = \sqrt{64 - 81((x-0.5)^2 + (y-0.5)^2)}/9 - 0.5.$

Table 1 contains the maximum errors for approximating the functions (3.1) by the classical and the modified Shepard operators given, respectively, by (1.1) and (1.3), and the errors of approximating by the operators introduced in (2.1), (2.3), (2.5) and (2.6). We consider three sets of $N = 100$ random points for each function in $[0,1] \times [0,1]$, $k = 25$ knots, $\mu = 3$ and $N_w = 19$.

Remark 3.1. The approximants $S_2 f_i$, $S_4 f_i$, $i = 1, 2, 3$ have better approximation properties although the number of knot points is smaller than the number of knot points considered for the approximants $S_1 f_i$, $S_3 f_i$ $i = 1, 2, 3$, so this illustrates the benefits of the algorithm of choosing the representative set of points.

In Figures 2, 4, 6 we plot the graphs of f_1, f_2, f_3 and of the corresponding Shepard operators $S_1 f_i$, $S_2 f_i$, $S_3 f_i$ and $S_4 f_i$, $i = 1, 2, 3$, respectively.

In Figures 1, 3, 5 we plot the sets of the given points and the corresponding sets of the representative knot points.

TABLE 1. Maximum approximation errors.

|	f_1	f_2	f_3
$S_\mu f$	0.0864	0.1095	0.1936
$S f$	0.0724	0.0970	0.1770
$S_1 f$	0.1644	0.4001	0.6595
$S_2 f$	0.1246	0.2858	0.3410
$S_3 f$	0.1578	0.3783	0.6217
$S_4 f$	0.1212	0.2834	0.3399

First set of given points.   First set of representative knot points.
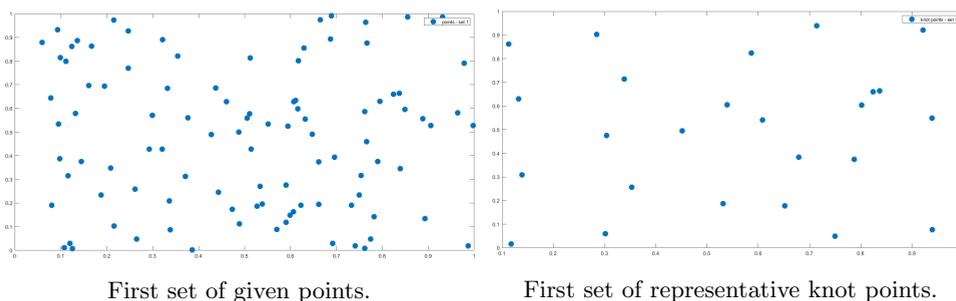
FIGURE 1. First sets of points.
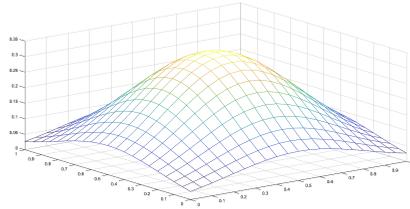
Function f_1.
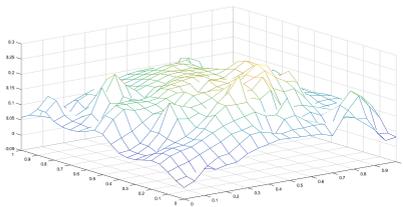
$S_1 f_1$
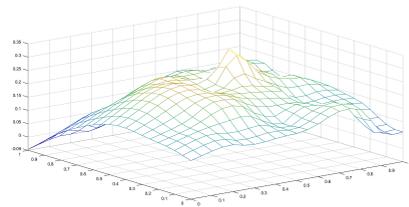
$S_2 f_1$
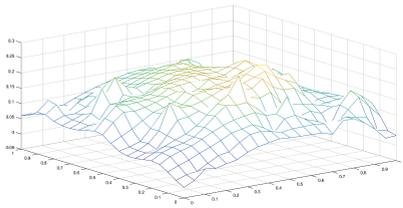
$S_3 f_1$
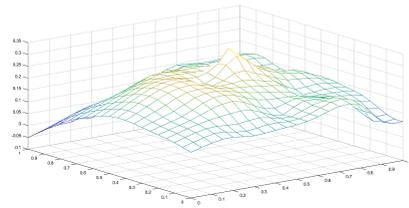
$S_4 f_1$

FIGURE 2. Graphs for f_1.
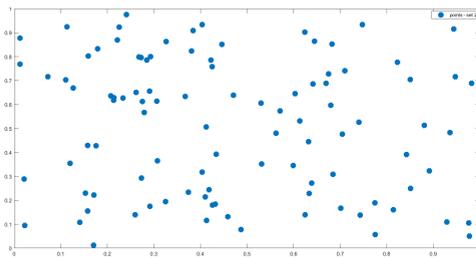
Second set of given points.
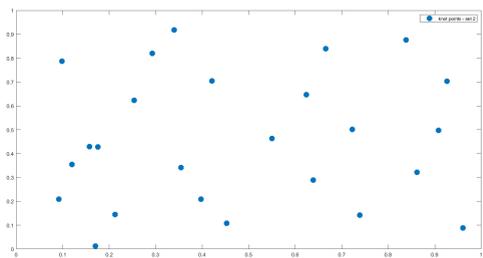
Second set of representative knot points.

FIGURE 3. Second sets of points.
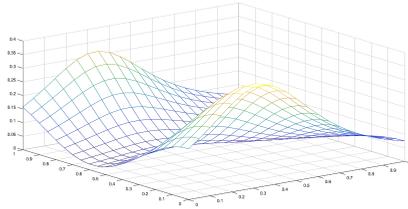
Function f_2.
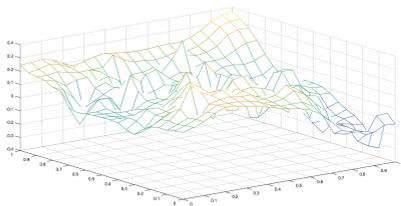
$S_1 f_2$
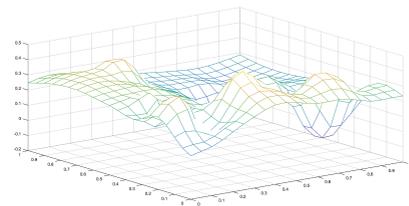
$S_2 f_2$
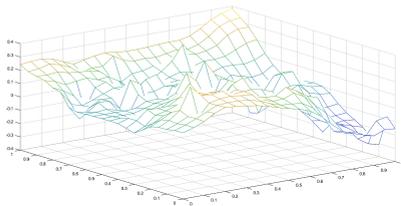
$S_3 f_2$
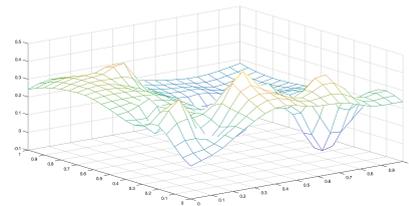
$S_4 f_2$

FIGURE 4. Graphs for f_2.
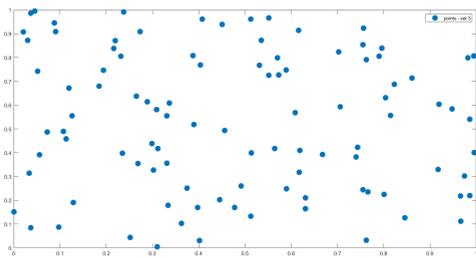
Third set of given points.
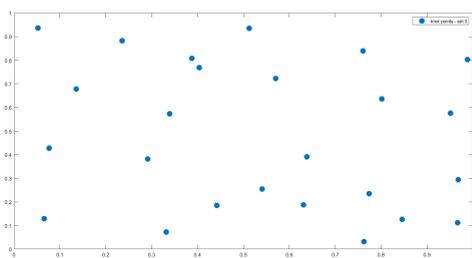
Third set of representative knot points.

FIGURE 5. Third sets of points.
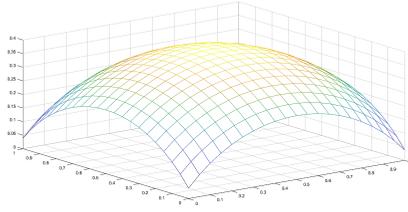
Function f_3.
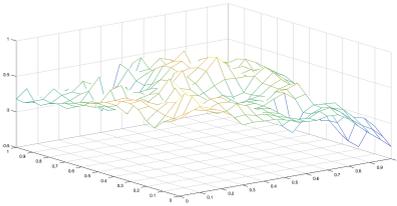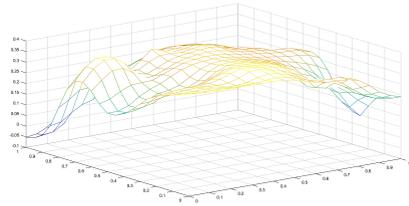
S_1f_3 S_2f_3
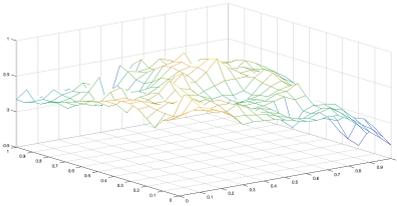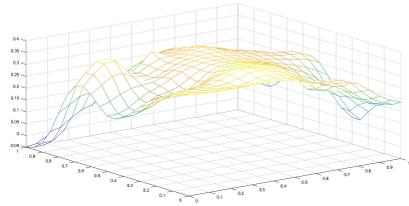
S_3f_3 S_4f_3

FIGURE 6. Graphs for f_3.

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