Stud. Univ. Babeş-Bolyai Math. 65(2020), No. 1, 67–75
DOI: 10.24193/subbmath.2020.1.06

Some properties of a linear operator involving generalized Mittag-Leffler function

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Abstract. This paper introduces a new class $T^{\gamma}_{\alpha,\beta,k}(\eta)$ of analytic functions which is defined by means of a linear operator involving generalized Mittag-Leffler function $\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)$. The results investigated in this paper include, an inclusion relation for functions in the class $T^{\gamma}_{\alpha,\beta,k}(\eta)$ and also some subordination results of the linear operator $\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)$. Several consequences of our results are also pointed out.

Mathematics Subject Classification (2010): 33E12, 30C45.

Keywords: Analytic functions, univalent functions, Mittag-Leffler function, differential subordination, convex function.

1. Introduction

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let f and g be analytic functions in $\mathbb{U}$, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function w on $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ for all $z \in \mathbb{U}$. In particular, if g is univalent in $\mathbb{U}$, then we have

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $E_{\alpha}(z)$ be the Mittag-Leffler function [11] defined by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (z, \alpha \in \mathbb{C}; \mathrm{Re}(\alpha) > 0\ ). \tag{1.1}$$

A more general function $E_{\alpha,\beta}$ generalizing $E_\alpha(z)$ was introduced by Wiman [14] and defined by

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad (z,\alpha,\beta\in\mathbb{C};\mathrm{Re}(\alpha)>0\ ). \tag{1.2}$$

Moreover, Srivastava and Tomovski [13] introduced the function $E_{\alpha,\beta}^{\gamma,k}(z)$ as

$$E_{\alpha,\beta}^{\gamma,k}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{nk}z^n}{\Gamma(\alpha n+\beta)n!},(\alpha,\beta,\gamma\in\mathbb{C};\mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\};\mathrm{Re}(k)>0),$$

where $(\gamma)_n$ is Pochhammer symbol (or the shifted factorial, since $(1)_n=n!$) is given in term of the Gamma functions can be written as

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}=\begin{cases}1, & \text{if } n=0;\\ \gamma(\gamma+1)...(\gamma+n-1), & \text{if } n\in\mathbb{N}.\end{cases} \tag{1.3}$$

The Mittag-Leffler function arises naturally in the solution of fractional order differential and integral equations, and especially in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g. in [2, 3, 4, 6, 7, 8, 9, 11, 12, 13].

In [1], Attiya defined the operator $\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f):\mathcal{A}\to\mathcal{A}$ by

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)=Q_{\alpha,\beta,k}^{\gamma}(z)*f(z),\qquad (z\in\mathbb{U}),$$

where

$$Q_{\alpha,\beta,k}^{\gamma}(z)=\frac{\Gamma(\alpha+\beta)}{(\gamma)_k}\left(E_{\alpha,\beta}^{\gamma,k}(z)-\frac{1}{\Gamma(\beta)}\right),\qquad (z\in\mathbb{U}),$$
$$(\alpha,\beta,\gamma\in\mathbb{C};\mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(k)-1\};\mathrm{Re}(k)>0;$$
$$\mathrm{Re}(\alpha)=0 \text{ when } \mathrm{Re}(k)=1 \text{ with } \beta\neq 0),$$

and the symbol $(*)$ denotes the Hadamard product (or convolution).
We note that,

$$\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\gamma+nk)\Gamma(\alpha+\beta)}{\Gamma(\gamma+k)\Gamma(\beta+\alpha n)n!}a_nz^n. \tag{1.4}$$

It can be easily verified from (1.4) that

$$z\left(\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)\right)'=\left(\frac{\gamma+k}{k}\right)(\mathcal{H}_{\alpha,\beta,k}^{\gamma+1}(f)(z))-\frac{\gamma}{k}(\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)). \tag{1.5}$$

Also we have

$$\mathcal{H}_{0,\beta,1}^{1}(f)(z)=f(z),\mathcal{H}_{0,\beta,1}^{2}(f)(z)=\frac{1}{2}\left(f(z)+zf'(z)\right) \text{ and } \mathcal{H}_{0,\beta,1}^{0}(f)(z)=\int_0^z\frac{1}{t}f(t)dt.$$

Definition 1.1. We say that the function $f \in \mathcal{A}$ is in the class $T^{\gamma}_{\alpha,\beta,k}(\eta), \eta \in [0,1)$, if f satisfies the condition

$$\mathrm{Re}\left[\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right]' > \eta, \qquad (z \in \mathbb{U}). \tag{1.6}$$

The object of this paper is to investigate an inclusion relation for functions in the class $T^{\gamma}_{\alpha,\beta,k}(\eta)$ and obtain some subordination results for functions defined by the linear operator $\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)$. Several consequences of our results are also discussed.

The following results will be required in our investigation.

Lemma 1.2. ([5]) *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is analytic in $\mathbb{U}$ and $h(z)$ is convex function in $\mathbb{U}$ with $h(0) = 1$ and μ is a complex constant such that* $\mathrm{Re}\mu > 0$, *then*

$$p(z) + \frac{zp'(z)}{\mu} \prec h(z), \tag{1.7}$$

implies

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\mu}{z^{\mu}} \int_0^z h(t) t^{\mu-1} dt,$$

and $q(z)$ is the best dominant.

Lemma 1.3. ([10]) *Let q be a convex function in $\mathbb{U}$ and let*

$$h(z) = q(z) + \alpha z q'(z),$$

where $\alpha > 0$. If

$$p(z) = q(0) + p_1 z + \cdots$$

and

$$p(z) + \alpha z p'(z) \prec h(z),$$

then

$$p(z) \prec q(z),$$

and this result is sharp.

2. Inclusion relation

We begin by showing the following inclusion relation.

Theorem 2.1. *If $\eta \in [0,1)$, then*

$$T^{\gamma+1}_{\alpha,\beta,k}(\eta) \subset T^{\gamma}_{\alpha,\beta,k}(\delta), \tag{2.1}$$

where

$$\delta = \delta(\eta,\gamma,k) = 2\eta - 1 + \frac{2(1-\eta)(\gamma+k)}{k} \mathbf{B}\left(\frac{\gamma+k}{k}\right), \tag{2.2}$$

B *being the Beta function defined by*

$$\mathbf{B}(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt. \tag{2.3}$$

Proof. Let $f \in T_{\alpha,\beta,k}^{\gamma+1}(\eta)$ and define the function $p(z)$ by

$$p(z) = \left(\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)\right)'. \tag{2.4}$$

Making use the identity (1.5), we get

$$\left(\mathcal{H}_{\alpha,\beta,k}^{\gamma+1}(f)(z)\right)' = p(z) + \frac{k}{\gamma+k} zp'(z), \qquad (z \in \mathbb{U}). \tag{2.5}$$

Since $f \in T_{\alpha,\beta,k}^{\gamma+1}(\eta)$, from Definition 1.1 we have

$$\operatorname{Re}\left(\mathcal{H}_{\alpha,\beta,k}^{\gamma+1}(f)(z)\right)' > \eta, \qquad (z \in \mathbb{U}).$$

Using (2.5) we get

$$\operatorname{Re}\left(p(z) + \frac{k}{\gamma+k} zp'(z)\right) > \eta,$$

which is equivalent to

$$p(z) + \frac{k}{\gamma+k} zp'(z) \prec \frac{1+(2\eta-1)z}{1+z} \equiv h(z).$$

By using Lemma 1.2, with $\mu = \frac{\gamma+k}{k}$ we have

$$p(z) \prec q(z) \prec h(z),$$

where

$$\begin{aligned} q(z) &= \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z \frac{1+(2\eta-1)t}{1+t} t^{\frac{\gamma+k}{k}-1} dt \\ &= \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z \left[2\eta-1+2(1-\eta)\right] \frac{1}{1+t} t^{\frac{\gamma+k}{k}-1} dt \\ &= \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z (2\eta-1) t^{\frac{\gamma+k}{k}-1} dt + \frac{2(1-\eta)(\gamma+k)}{kz^{\frac{\gamma+k}{k}}} \int_0^z \frac{t^{\frac{\gamma+k}{k}-1}}{1+t} dt \\ &= 2\eta-1+\frac{2(1-\eta)(\gamma+k)}{kz^{\frac{\gamma+k}{k}}} \int_0^z \frac{t^{\frac{\gamma+k}{k}-1}}{1+t} dt. \end{aligned}$$

The function q is convex and is the best dominant.
Since $p(z) \prec q(z)$, we get

$$\operatorname{Re}\left[\mathcal{H}_{\alpha,\beta,k}^{\gamma}(f)(z)\right]' > q(1) = \delta, \tag{2.6}$$

where

$$\delta = \delta(\eta, \gamma, k) = 2\eta - 1 + \frac{2(1-\eta)(\gamma+k)}{k}\mathbf{B}\left(\frac{\gamma+k}{k}\right).$$

From (2.6) we deduce that $T^{\gamma+1}_{\alpha,\beta,k}(\eta) \subset T^{\gamma}_{\alpha,\beta,k}(\delta)$. □

3. Subordination results

With the help of Lemma 1.3, we obtain the following result.

Theorem 3.1. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + \frac{k}{\gamma+k} zq'(z). \tag{3.1}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \tag{3.2}$$

then

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \tag{3.3}$$

and the result is sharp.

Proof. From (2.5) and (3.2) we obtain

$$p(z) + \frac{k}{\gamma+k} zp'(z) \prec q(z) + \frac{k}{\gamma+k} zq'(z) \equiv h(z),$$

then, by using Lemma 1.3 we get

$$p(z) \prec q(z),$$

that is,

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \qquad (z \in \mathbb{U}),$$

and this result is sharp. □

Theorem 3.2. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \qquad (z \in \mathbb{U}). \tag{3.4}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \tag{3.5}$$

then

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \tag{3.6}$$

where

$$q(z) = \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z h(t) t^{\frac{\gamma+k}{k}-1} dt.$$

The function q is convex and is the best dominant.

Proof. If we let

$$p(z) = \left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)',$$

and using the identity (1.5), we obtain

$$\left(\mathcal{H}^{\gamma+1}_{\alpha,\beta,k}(f)(z)\right)' = p(z) + \frac{k}{\gamma+k} zp'(z), \qquad (z \in \mathbb{U}).$$

Therefore, (3.5) becomes

$$p(z) + \frac{k}{\gamma+k} zp'(z) \prec h(z).$$

By using Lemma 1.2, we get

$$p(z) \prec q(z) = \frac{\gamma+k}{kz^{\frac{\gamma+k}{k}}} \int_0^z h(t)t^{\frac{\gamma+k}{k}-1} dt,$$

that is,

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec q(z), \qquad (z \in \mathbb{U}).$$

□

Theorem 3.3. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$. And let h be a function such that*

$$h(z) = q(z) + zq'(z), \qquad (z \in \mathbb{U}). \tag{3.7}$$

If $f \in A$ and verifies the differential subordination

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \tag{3.8}$$

then

$$\frac{\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)}{z} \prec q(z), \tag{3.9}$$

and the result is sharp.

Proof. Let the function $p(z)$ be defined by

$$p(z) = \frac{\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)}{z}. \tag{3.10}$$

Then, by differentiating (3.10), we get

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' = p(z) + zp'(z), \qquad (z \in \mathbb{U}). \tag{3.11}$$

Thus (3.8) becomes

$$p(z) + zp'(z) \prec q(z) + zq'(z) \equiv h(z),$$

and from Lemma 1.3 we get (3.9). □

Theorem 3.4. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality (3.4). If $f \in \mathcal{A}$ and verifies the differential subordination*

$$\left(\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)\right)' \prec h(z), \qquad (z \in \mathbb{U}), \tag{3.12}$$

then

$$\frac{\mathcal{H}^{\gamma}_{\alpha,\beta,k}(f)(z)}{z} \prec q(z), \qquad (z \in \mathbb{U}, z \neq 0), \tag{3.13}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is convex and is the best dominant.

Proof. Let the function $p(z)$ be defined as in (3.10). Then from (3.11) and (3.12),we have

$$p(z) + zp'(z) \prec h(z).$$

By using Lemma 1.2, we get

$$p(z) \prec q(z) = \frac{1}{z}\int_0^z h(t)dt,$$

and q is convex and is the best dominant. □

If we set $\gamma = 1$, $\alpha = 0$ and $k = 1$, in Theorems 3.1-3.4, we immediately have the following special cases.

Corollary 3.5. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + \frac{1}{2}zq'(z). \tag{3.14}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$f'(z) + \frac{1}{2}zf''(z) \prec h(z), \tag{3.15}$$

then

$$f'(z) \prec q(z), \tag{3.16}$$

and the result is sharp.

Corollary 3.6. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality (3.4). If $f \in \mathcal{A}$ and verifies the differential subordination*

$$f'(z) + \frac{1}{2}zf''(z) \prec h(z), \tag{3.17}$$

then

$$f'(z) \prec q(z), \tag{3.18}$$

where

$$q(z) = \frac{2}{z^2}\int_0^z h(t)tdt.$$

The function q is convex and is the best dominant.

Corollary 3.7. *Let $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + zq'(z), \qquad (z \in \mathbb{U}). \tag{3.19}$$

If $f \in \mathcal{A}$ and verifies the differential subordination

$$f'(z) \prec h(z), \tag{3.20}$$

then

$$\frac{f(z)}{z} \prec q(z), \tag{3.21}$$

and the result is sharp.

Corollary 3.8. *Let $h \in \mathcal{A}$ with $h(0) = 1$ and $h'(0) \neq 0$, which verifies the inequality (3.4). If $f \in \mathcal{A}$ and verifies the differential subordination*

$$f'(z) \prec h(z), \qquad (z \in \mathbb{U}), \tag{3.22}$$

then

$$\frac{f(z)}{z} \prec q(z), \qquad (z \in \mathbb{U}, z \neq 0), \tag{3.23}$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt.$$

The function q is convex and is the best dominant.

Acknowledgements. The authors would like to thank the referee for his helpful comments and suggestions.

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