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NUMERICAL GENERATION OF SYMMETRIC α-STABLE RANDOM VARIABLES

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Abstract. The paper discusses two extensions to higher order of the fast, accurate algorithm due to Mantegna [9] for the numerical generation of symmetric α-stable random variables. These extensions result in improved computing time over the most usual range of the index of stability, $\alpha > 1$, for which expectations exist.

1. Introduction

Lévy processes are a class of stochastic processes which enjoy a rich mathematical structure and are increasingly used in applications ranging from finance [3] to the study of non-Fickian diffusion in physical systems [8]. Since exact solutions to stochastic differential equations (SDEs) driven by Lévy noise are not usually available, the numerical approximation of such SDEs is often needed. When the path of the Lévy process has to be constructed explicitly, i.e. in the case of strong approximation, the numerical generation of a large number of random variables with the corresponding Lévy distribution is necessary. Even more so, in numerical approximations of some integro-differential nonlinear partial equations of evolution based on the interacting particles approximation [14], the position of each particle is governed by a SDE driven by Lévy noise, hence a system of SDEs of size equal to the number of particles must be integrated numerically. In such a case, the use of a fast and accurate algorithm for the generation of these random variables (which represent discrete approximations to the time increments of the stochastic process) is crucial if reliable numerical results are to be obtained in a convenient time frame.

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A first numerical algorithm for the generation of random variables with a general Lévy distribution, including those with skewness, has been presented by Chambers, Mallows and Stuck [4]. More recently, Mantegna [9] devised a different numerical method for the class of symmetric α-stable Lévy distributions based on the asymptotic expansion of the integral expression of their probability density function. Mantegna's algorithm makes use of the generalized version of the central limit theorem together with a nonlinear transformation to achieve an accurate approximation of the probability density function. However, the use of the generalized central limit theorem by this latter algorithm implies summation of several independent realizations of a random variable with a probability distribution close to the targeted distribution. Although the number of these samples is reduced by a nonlinear transformation, the generation of several independent samples reduces the efficiency of the algorithm.

In this paper, we propose a new algorithm for the numerical generation of α-stable random variables. It is based, as Mantegna's algorithm [9], on the asymptotic expansion of the probability density function, but to the next higher order. The use of the higher-order term introduces some complications in the evaluation of the associated probabilities, which could not be surmounted analytically so that numerical approximations were needed. The paper is organized as follows. The next two sections briefly recall the basic properties of symmetrical Lévy α-stable random variables that we need and the algorithm due to Mantegna. Section 4 develops our proposed algorithm, while the last section presents pertinent numerical results comparing the algorithms. A brief conclusion section ends the paper.

2. Symmetric $\alpha-$stable distributions

We recall that a univariate random variable Z has a (strictly) stable distribution if for any $a, b > 0$ there exists $c > 0$ such that $aZ_1 + bZ_2 \stackrel{d}{=} cZ$, where Z_1 and Z_2 are independent copies of Z, and $\stackrel{d}{=}$ denotes equality in distribution. For given α, the distribution of Z is called symmetric α-stable if it equals the distribution of $-Z$, and in this case its probability density function (PDF) can be expressed as the improper integral

$$f_Z^{\alpha,\gamma}(z) = \frac{1}{\pi}\int_0^\infty \exp(-\gamma q^\alpha)\cos(qz)\,dq, \quad 0 < \alpha \le 2 \tag{1}$$

The parameter α is known as the index of stability, or characteristic exponent, of the distribution, while γ is a scale factor ($\gamma > 0$). For $\alpha = 2$, $\alpha = 1$ and $\alpha = 1/2$, the Gauss, Cauchy and Lévy distributions are obtained, respectively.

Humbert [5] discusses the problem of representing the derivative of e^{-q^α} as a Laplace integral. His result leads to the following expression for $f_Z^{\alpha,\gamma}(z)$:

$$f_Z^{\alpha,\gamma}(z) = -\frac{1}{\pi}\sum_{k=1}^{N}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{z^{\alpha k+1}}\sin\left(\frac{k\pi\alpha}{2}\right) + R(z) \tag{2}$$

where $\Gamma(z)$ is the gamma function and $R(z) = O\left(z^{-\alpha(N+1)-1}\right)$. From (2), one can obtain the two-term asymptotic approximation of a symmetric stable PDF for large z as a function of the parameter α,

$$f_Z^{\alpha,\gamma}(z) \approx \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi z^{1+\alpha}} - \frac{\Gamma(1+2\alpha)\sin(\pi\alpha)}{\pi z^{1+2\alpha}} \tag{3}$$

For more details about stable distributions we refer to [6, 13].

3. Computer generation of symmetric α-stable random variables

While the work of Chambers *et al.* [4] describes a method for generation of α-stable random variables with general distributions that may include skewness, a different approach valid for the symmetric α-stable case was taken by Mantegna [9]. The latter results in an algorithm allowing the generation of a random variable Z whose probability density is $a$rbitrarily close to the PDF (1) for $0.3 \le \alpha \le 1.99$. The main idea stems from the generalized central limit theorem: the sum of independent random variables having the same symmetric α-stable distribution will eventually converge to a random variable characterized by the same law. Given α, consider the random variable

$$V = \frac{X}{|Y|^{1/\alpha}}, \tag{4}$$

where X and Y are two normal random variables with standard deviation σ_x and σ_y, respectively. One can then choose these values such that the probability density function of V, $f_V(v)$ matches the exact PDF $f_Z^{\alpha,1}(z)$ in the origin and for large values of z. To obtain better results, one can then generate a number of independent copies of V, say $V_1, V_2, \ldots, V_n$, and use the central limit theorem to construct

$$\tilde{Z} = \frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} V_k. \tag{5}$$

The random variable $\tilde{Z}$ may be expected to converge to a symmetric α-stable random variable. Because the convergence is quite slow, i.e. one needs a relatively large n in equation (5), Mantegna introduced a nonlinear transformation which gives an exponential tilt to the distribution of the random variable V by defining a new random variable,

$$W = \{[K(\alpha) - 1][\exp(-|V|/C(\alpha))] + 1\} V, \tag{6}$$

with parameters $K(\alpha)$ and $C(\alpha)$ determined by requiring

$$P(W = 0) = f_Z^{\alpha,1}(0) \tag{7}$$

and respectively

$$P[W = W(C(\alpha))] = f_Z^{\alpha,1}[W(C(\alpha))]. \tag{8}$$

A fast convergence toward a stable random variable is then obtained by constructing

$$\tilde{Z} = \frac{1}{n^{1/\alpha}} \sum_{k=1}^{n} W_k, \tag{9}$$

instead of (5). Note that the cost of Mantegna's algorithm depends on the number n of samples of the random variable W used in equation (9). Larger values of n make the algorithm more accurate at the price of generating many copies of W, each of which requires two samples from a normal distribution.

4. **High-Order Algorithm Using Independent Samples**

In the following, we propose a new algorithm for the numerical generation of a symmetric α-stable random variable which has the same starting point as Mantegna's algorithm [9], but is much faster for comparable accuracy. Note that one can set $\gamma = 1$ for simplicity, since rescaling of the generated random variable is straightforward.

First, consider equation (2) with $N = 2$ and $\gamma = 1$ and let us compute

$$V_1 = \frac{X_1}{|Y_1|^{1/\alpha}} \text{ and } V_2 = \frac{X_2}{|Y_2|^{1/2\alpha}}, \tag{10}$$

where X_1, X_2, Y_1, Y_2 are four independent normal random variables with standard deviation $\sigma_{x_1}, \sigma_{x_2}, \sigma_{y_1}, \sigma_{y_2}$ respectively. Using the method of transformations for the bivariate case, see e.g. [12], the probability densities of the continuous variables V_1 and V_2 can be found to be

$$
\begin{aligned}
f_{V_1}(v_1) &= \frac{1}{\pi\sigma_{x_1}\sigma_{y_1}} \int_0^\infty y^{1/\alpha} \exp\left[-\frac{y^2}{2\sigma_{y_1}^2} - \frac{v_1^2 y^{2/\alpha}}{2\sigma_{x_1}^2}\right] dy, \\
f_{V_2}(v_2) &= \frac{1}{\pi\sigma_{x_2}\sigma_{y_2}} \int_0^\infty y^{1/2\alpha} \exp\left[-\frac{y^2}{2\sigma_{y_2}^2} - \frac{v_2^2 y^{1/\alpha}}{2\sigma_{x_2}^2}\right] dy,
\end{aligned}
\tag{11}
$$

For large arguments, the above probability densities are very well described by the asymptotic approximation

$$
\begin{aligned}
f_{V_1}(v_1 \gg 0) &\approx \frac{\alpha 2^{(\alpha-1)/2}\sigma_{x_1}^{\alpha}\Gamma((\alpha+1)/2)}{\pi\sigma_{y_1} v_1^{\alpha+1}}, \\
f_{V_2}(v_2 \gg 0) &\approx \frac{\alpha 2^{(2\alpha+1)/2}\sigma_{x_2}^{2\alpha}\Gamma((2\alpha+1)/2)}{\pi\sigma_{y_2} v_2^{2\alpha+1}}
\end{aligned}
\tag{12}
$$

and in the origin

$$
\begin{aligned}
f_{V_1}(0) &= \frac{2^{(1-\alpha)/2\alpha}\sigma_{y_1}^{1/\alpha}\Gamma((\alpha+1)/2\alpha)}{\pi\sigma_{x_1}}, \\
f_{V_2}(0) &= \frac{2^{(1-2\alpha)/4\alpha}\sigma_{y_2}^{1/2\alpha}\Gamma((2\alpha+1)/4\alpha)}{\pi\sigma_{x_2}}.
\end{aligned}
\tag{13}
$$

The second step in our algorithm is to compute another random variable V given by

$$
V = V_1 + V_2. \tag{14}
$$

The density of the sum of two independent continuous random variables is the convolution of their individual densities. Considering, without loss of generality, the particular case where $\sigma_{y_1} = \sigma_{y_2} = 1$, it follows then [12] that the probability density of the random variable V is given by

$$
\begin{aligned}
f_V(v) \approx \int_{-\infty}^{\infty} &\left[\int_0^\infty \frac{1}{\pi\sigma_{x_1}} s^{\frac{1}{\alpha}} \exp\left(-\frac{s^2}{2} - \frac{t^2}{2\sigma_{x_1}} s^{\frac{2}{\alpha}}\right) ds\right] \cdot \\
&\cdot \left[\int_0^\infty \frac{1}{\pi\sigma_{x_2}} s^{\frac{1}{2\alpha}} \exp\left(-\frac{s^2}{2} - \frac{(v-t)^2}{2\sigma_{x_2}} s^{\frac{1}{\alpha}}\right) ds\right] dt
\end{aligned}
\tag{15}
$$

hence its value at the origin is

$$f_V(0) = \int_{-\infty}^{\infty} \left[\int_0^{\infty} \frac{1}{\pi\sigma_{x_1}} s^{\frac{1}{\alpha}} \exp\left(-\frac{s^2}{2} - \frac{t^2}{2\sigma_{x_1}} s^{\frac{2}{\alpha}} \right) ds \right] \cdot \left[\int_0^{\infty} \frac{1}{\pi\sigma_{x_2}} s^{\frac{1}{2\alpha}} \exp\left(-\frac{s^2}{2} - \frac{t^2}{2\sigma_{x_2}} s^{\frac{1}{\alpha}} \right) ds \right] dt \tag{16}$$

We now obtain values for σ_{x_1} and σ_{x_2} such that the following conditions are satisfied simultaneously for a given value of α:

- The approximate PDF matches the exact one in the origin,

$$f_Z^{\alpha,1}(0) = f_V(0) \tag{17}$$

- The least-squares error in the approximate PDF is minimized over a bounded interval $[-L, L]$:

$$F(\sigma_{x_1}, \sigma_{x_2}) = \int_{-L}^{L} [f_Z^{\alpha,1}(z) - f_V(z)]^2 dz = \min . \tag{18}$$

From these conditions one obtains a system of equations that can be solved numerically for the values of $\sigma_{x_1}, \sigma_{x_2}$ once α and a value for L are specified.

5. **High-Order Algorithm Using Dependency**

Another approach which is less computationally expensive but involves some tedious, albeit straightforward algebraic manipulation, is to reduce the number of independent normal variables generated in the high-order algorithm. This can be done as follows. Note that in (10) four independent normal random variables are used, although there are only two free unknowns. To further reduce the computational cost, let (10) hold for $X_1 = X_2 = X$ and $Y_1 = Y_2 = Y$, where X, Y are two independent normal random variables with standard deviation σ_x, σ_y respectively. Hence (10) is equivalent to

$$V_1 = \frac{X}{|Y|^{1/\alpha}} \text{ and } V_2 = \frac{X}{|Y|^{1/2\alpha}}. \tag{19}$$

With this choice, the random variables V_1 and V_2 are dependent, and the joint density of V_1 and V_2 becomes more difficult to evaluate. The method of transformations [12] for the bivariate case will be used again to compute the probability density function

$f_{V_1,V_2}(v_1,v_2)$ of (V_1,V_2).
Let $g(x,y)=\left(\frac{x}{|y|^{1/\alpha}},\frac{x}{|y|^{1/2\alpha}}\right)=(v_1,v_2)$. Note that

$$g^{-1}(v_1,v_2)=\left(\frac{v_2^2}{v_1},\left(\frac{v_2}{v_1}\right)^{2\alpha}\right)=\left(\frac{x^2}{|y|^{1/\alpha}}\cdot\frac{|y|^{1/\alpha}}{x},\frac{x^{2\alpha}}{|y|}\cdot\frac{|y|^2}{x^{2\alpha}}\right)=(x,|y|).$$

Therefore, $g^{-1}(v_1,v_2)=\left(\frac{v_2^2}{v_1},\left(\frac{v_2}{v_1}\right)^{2\alpha}\right)$. The absolute value of the determinant of the Jacobian $J_{g^{-1}}$ is given by

$$\begin{aligned}|J_{g^{-1}}(v_1,v_2)| &= abs\begin{vmatrix}\frac{\partial x}{\partial v_1} & \frac{\partial x}{\partial v_2}\\ \frac{\partial y}{\partial v_1} & \frac{\partial y}{\partial v_2}\end{vmatrix}=abs\begin{vmatrix}-\frac{v_2^2}{v_1^2} & 2\frac{v_2}{v_1}\\ v_2^{2\alpha}(-2\alpha v_1^{-2\alpha-1}) & 2\alpha v_2^{2\alpha-1}v_1^{-2\alpha}\end{vmatrix}\\ &=|(-2\alpha v_2^{2\alpha+1}v_1^{-2\alpha-2}+4\alpha v_2^{2\alpha+1}v_1^{-2\alpha-2}|=2\alpha|(v_2^{2\alpha+1}v_1^{-2\alpha-2})|.\end{aligned}$$

Hence, the probability density function $f_{V_1,V_2}(v_1,v_2)$ of (V_1,V_2) is given by

$$f_{V_1,V_2}(v_1,v_2)=2\alpha|v_2^{2\alpha+1}v_1^{-2\alpha-2}|[f_X(\frac{v_2^2}{v_1})f_Y((\frac{v_2}{v_1})^{2\alpha})+f_X(\frac{v_2^2}{v_1})f_Y(-(\frac{v_2}{v_1})^{2\alpha})].$$

Next, let $\{V\}$ to be another random variable given by

$$V=V_1+V_2. \tag{20}$$

The probability density function of the random variable V is given by (see [12])

$$\begin{aligned}f_V(v) &= \int_{-\infty}^{\infty} f_{V_1,V_2}(w,v-w)dw\\ &= 4\alpha\tfrac{1}{2\pi\sigma_x\sigma_y}\int_{-\infty}^{\infty}|(v-w)^{2\alpha+1}w^{-2\alpha-2}|\exp\left(-\tfrac{1}{2\sigma_x^2}\tfrac{(v-w)^4}{w^2}-\tfrac{1}{2\sigma_y^2}\tfrac{(v-w)^{4\alpha}}{w^{4\alpha}}\right)dw.\end{aligned} \tag{21}$$

In order to obtain values for σ_x and σ_y for a given value of α, one can again impose conditions similar to those stated in equations (17) and (18). Lastly, let us note that this use of dependent variables reduces the cost of the algorithm in the previous section by a factor of two.

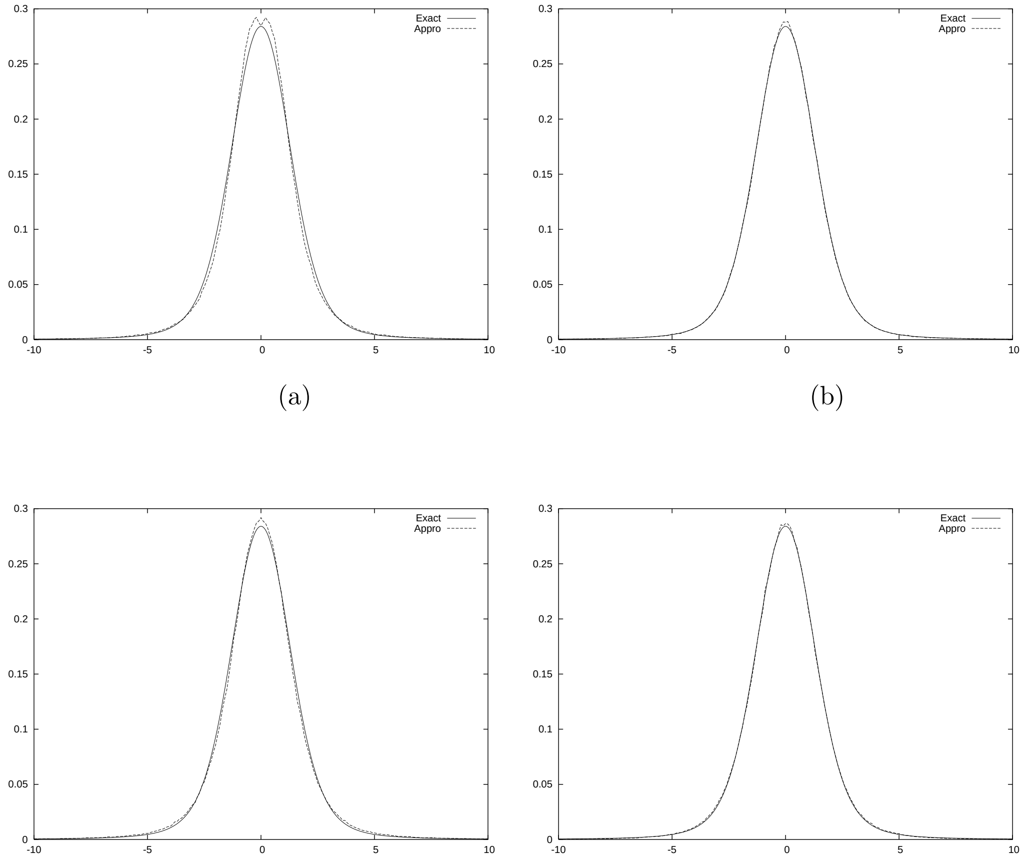
6. **Numerical tests**

Table 1 gives a sample set of values obtained for the two parameters σ_{x_1} and σ_{x_2} (independent case) as a function of α, with the choice $L=10$.

α	$\sigma_{x_1}(\alpha)$	$\sigma_{x_2}(\alpha)$
0.7	0.880	0.002
0.8	0.930	0.001
0.9	0.971	0.000
1	1	0.027
1.1	0.951	0.215
1.2	0.855	0.410
1.3	0.800	0.523
1.4	0.729	0.645
1.5	0.610	0.800
1.6	0.396	1.008
1.7	0.280	1.100
1.8	0.110	1.200
1.9	0.001	1.231

TABLE 1. Values obtained for the parameters σ_{x_1} and σ_{x_2} as a function of α.

Probability density functions obtained numerically by the proposed algorithms, as well as by the algorithm due to Mantegna for both $n = 1$ and $n = 10$ in equation (9) are compared with the exact density in figures 1 and 2 for $\alpha = 1.7$ and $\alpha = 1.3$, respectively. For completeness, we also include results obtained with the corrected version of the algorithm due to Chambers *et al.* [4, 13]. In these figures the dashed lines are the result of the simulation (histograms based on 10^6 samples), while the continuous line is the exact Lévy stable distribution, computed from the integral form evaluated with 20 decimal digits in the symbolic computation package **Maple**, see `http://www.maplesoft.com`. The L_2 error in the numerically generated probability distributions as a function of α, again based on 10^6 samples, is given in figure 3, with the actual CPU time needed shown in figure 4. As can be seen, under this measure, our algorithm offers an accuracy comparable to Mantegna's method with $n = 10$ for a much smaller computational cost.



(a) (b)

(c) (d)

FIGURE 1. Exact and approximate Lévy density with $N = 1,000,000$ samples for $\alpha = 1.7$.
(a) Mantegna ($n = 1$)
(b) Mantegna ($n = 10$)
(c) Chambers *et al.* (d) Present

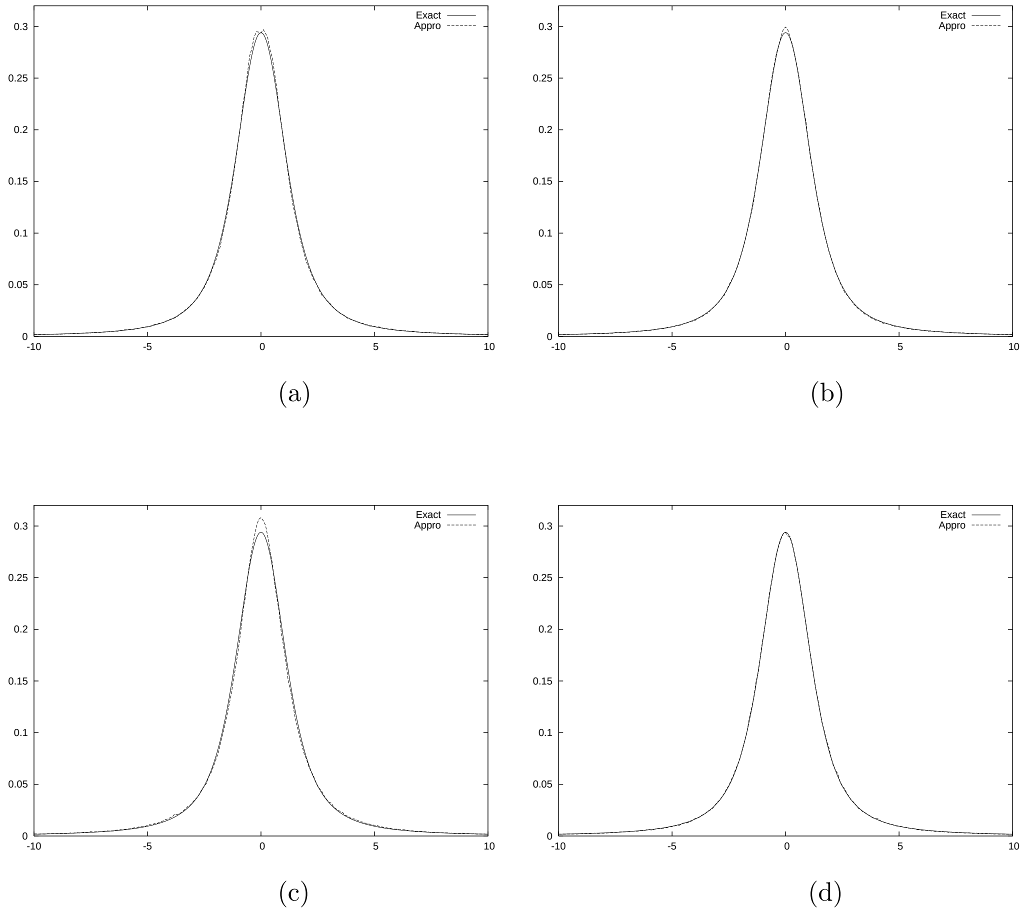


Figure 2. Exact and approximate Lévy density with $N = 1,000,000$ samples for $\alpha = 1.3$.
(a) Mantegna ($n = 1$)
(b) Mantegna ($n = 10$)
(c) Chambers *et al.*
(d) Present

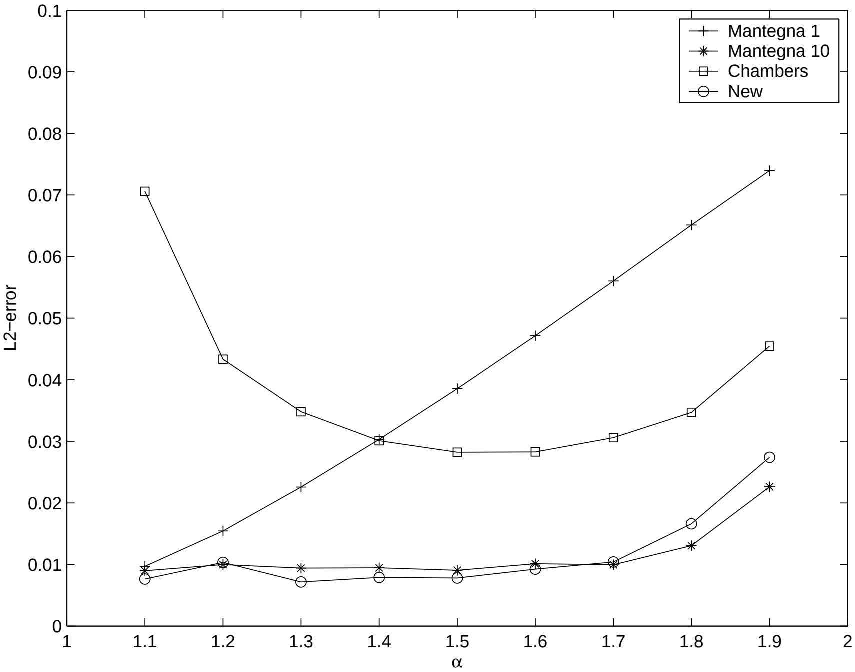


FIGURE 3. L_2 error as a function of α for Mantegna ($n = 1$ and $n = 10$), Chambers, and the proposed algorithm.

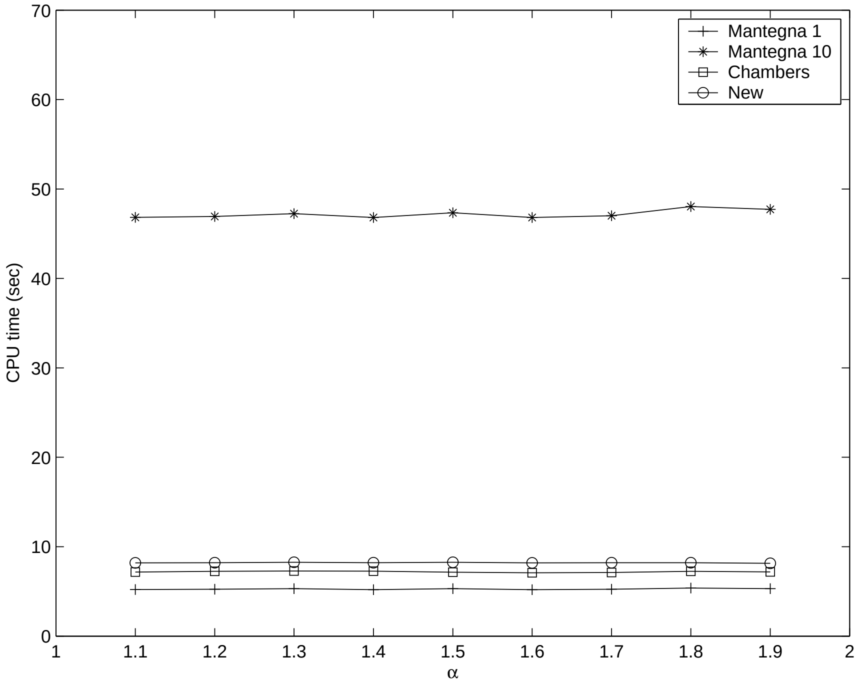


FIGURE 4. CPU time required for 10^6 samples using Mantegna's method ($n = 1$ and $n = 10$), Chambers, and the proposed algorithm.

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