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INDECOMPOSABLE SUBGROUPS OF TORSION-FREE ABELIAN GROUPS

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Dedicated to Professor Grigore Călugăreanu on his 60th birthday

Abstract. The present work gives descriptions of some classes of torsion-free abelian groups which have indecomposable subgroups, and for such a group we will present the structure of these subgroups.

All through this paper by group we mean abelian group in additive notation, and we will mark with P the set of all prime numbers, with $r(A)$ - the rank of the group A and with $t(A)$ - the type of A. According to [2,27.4], if a group A is indecomposable, then it is either torsion-free or cocyclic. Since the case of cocyclic groups (so of torsion groups) is well-known, in this paper we will present certain classes of torsion-free abelian groups which have indecomposable subgroups, properly constructing these subgroups.

Let G be a group of rank r and let $L = \{x_i\}_{i\in I}$ be a maximal independent system in G; I is a index set with $|I| = r$. Then, according to [2,16.1], $\langle L\rangle = \bigoplus_{i\in I}\langle x_i\rangle$. For every $i \in I$, we mark with X_i the pure subgroup of G, generated by $\{x_i\}$. Then, for every $i \in I$, the subgroup X_i is (homogeneous) of rank one and $t(X_i) = t(x_i) = t_i$. Therefore any group of rank r has a completely decomposable subgroup of the same rank. This motivates the study of our problem for completely decomposable groups.

For the beginning we have:

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Theorem 1. *Let* $G = \bigoplus_{i \in I} G_i$ *be a torsion-free group with the following properties:*

1) I is at most countable index set;

2) For every $i \in I$, G_i *is a reduced group and* $r(G_i) = 1$;

3) There is at most countable set $P_0 = \{p_1, p_2, \dots, p_n, \dots\}$ *of distinct prime numbers, with the following properties:*

i) for every $p_k \in P_0$, G_k *is* p_k*-divisible,*

ii) there is $p_{l_k} \in P_0 \setminus \{p_k\}$ *such that* G_k *is not* p_{l_k}*-divisible.*

Then G has an indecomposable subgroup A with $r(A) = |P_0|$.

Proof. Let G be a group as in the statement. For every $p_k \in P_0$, we consider the groups $E_{p_k} = \langle p_k^{-\infty} g_k \rangle$ and $H_{p_k} = \langle p_k^{-\infty} g_k, p_{l_k}^{-1} \overline{g}_k \rangle$, where $p_{l_k} \in P_0 \setminus \{p_k\}$ and $g_k \in G_k \setminus \{0\}$. Then, for every $p_k \in P_0$, E_{p_k} is a subgroup in H_{p_k} and $r(E_{p_k}) = 1$. Let be $A = \langle \bigoplus_{p_k \in P_0} E_{p_k}, p_{l_1}^{-1}\overline{g}_1 + p_{l_2}^{-1}\overline{g_2}, p_{l_1}^{-1}\overline{g}_1 + p_{l_3}^{-1}\overline{g}_3, \dots, p_{l_1}^{-1}\overline{g}_1 + p_{l_n}^{-1}\overline{g}_n, \dots \rangle$. Then $A \leq \bigoplus_{p_k \in P_0} H_{p_k} \leq \bigoplus_{i \in I} G_i$ and, for every $p_k \in P_0$, neither g_k nor $\overline{g}_k$ is divisible by $p_{l_k} \in P_0 \setminus \{p_k\}$ in A. (1)

Since, for every $p_k \in P_0$, all elements of E_{p_k} are divisible by every power of p_k, and, for every $p_s \in P_0 \setminus \{p_k\}$, E_{p_s} does not contain any such except for 0, it follows that, for every $p_k \in P_0$, E_{p_k} is fully invariant in A. On the other hand, since, for every $p_k \in P_0$, H_{p_k}/E_{p_k} is torsion, it follows that $(\bigoplus_{p_k \in P_0} H_{p_k})/(\bigoplus_{p_k \in P_0} E_{p_k})$ is torsion too. According to [1,1.6.12], it follows that $\bigoplus_{p_k \in P_0} E_{p_k}$ is an essential subgroup of A. (2)

We are going to show that A is indecomposable. In this way we suppose that $A = B \oplus C$. Then, according to [2,9.3], for every $p_k \in P_0$, $E_{p_k} = (E_{p_k} \cap B) \oplus (E_{p_k} \cap C)$. Since each E_{p_k} is indecomposable, it follows that, for every $p_k \in P_0$, either $E_{p_k} \cap B = 0$ or $E_{p_k} \cap C = 0$. So, for every $p_k \in P_0$, either $E_{p_k} \subseteq B$ or $E_{p_k} \subseteq C$. We suppose that there is $k \neq 1$ such that $E_{p_1} \subseteq B$ and $E_{p_k} \subseteq C$. In this case we consider the element $p_{l_1}^{-1}\overline{g}_1 + p_{l_k}^{-1}\overline{g}_k = b + c$, with $b \in B$ and $c \in C$. It follows that $p_{l_k}\overline{g}_1 + p_{l_1}\overline{g}_k = p_{l_1}p_{l_k}(b + c)$, that is $p_{l_k}(\overline{g}_1 - p_{l_1}b) = p_{l_1}(\overline{g}_k - p_{l_k}c) = 0$, which is impossible, according to the statement (1). Therefore, for every $p_k \in P_0$, either $E_{p_k} \subseteq B$ in which case $C = 0$, or $E_{p_k} \subseteq C$ in which case $B = 0$, according to the statement (2). It follows that A is indecomposable and since $r(A) = |P_0|$, the theorem is completely proved.

Corollary 2. *If $G = \bigoplus_{i\in I} G_i$ is a group which satisfies the conditions from Theorem 1 and the sets I and P_0 are equipotent, then G has indecomposable subgroups of every rank $m \le r(G)$.*

Now we obtain the example 2 from [3,p.123]:

Corollary 3. *Let $G = \bigoplus_{i\in I} G_i$ be a group which satisfies the conditions from Theorem 1. If there is $q \in P \setminus P_0$ such that in the condition 3) of Theorem 1, for every $p_k \in P_0$, p_{l_k} may be replaced by q, then G has indecomposable subgroups of every rank $m \le |P_0|$.*

Proof. Keeping the notations from Theorem 1, for every cardinal $m \le |P_0|$, we consider the indecomposable subgroup $A_m = \langle \bigoplus_{p_k \in P_0^{(m)}} E_{p_k}, q^{-1}(\overline{g}_1 + \overline{g}_2), q^{-1}(\overline{g}_1 + \overline{g}_3), \ldots, q^{-1}(\overline{g}_1 + \overline{g}_m)\rangle$, where $P_0^{(m)}$ is a subset of cardinal m of P_0.

Other consequences of Theorem 1:

Corollary 4. *Let $G = \bigoplus_{p\in P} G_p$ be a torsion-free group with the following properties:*

1) For every $p \in P$, G_p is p-divisible and $r(G_p) = 1$;

2) For every $p \in P$, there is a $q_p \in P \setminus \{p\}$ for which G_p is not q_p-divisible.

Then G has indecomposable subgroups of every rank $m \le r(G)$.

Proof. Let G be a group as in the statement. According to the condition 1), for every $p \in P$, there is a $\overline{g}_p \in G_p$ such that $h_p^{G_p}(\overline{g}_p) = \infty$. From the condition 2) it follows that, for every $p \in P$, there is a $q_p \in P \setminus \{p\}$ for which there is a $g_p \in G_p$ such that $h_{q_p}^{G_p}(g_p) = 1$. Since $r(G_p) = 1$, it follows that $\overline{g}_p$ and g_p are linear dependent; so $h_p^{G_p}(g_p) = \infty$. Now, for every $p_k \in P$, we consider the groups $E_{p_k} = \langle p_k^{-\infty}, g_{p_k}\rangle$ and $H_{p_k} = \langle p_k^{-\infty} g_{p_k}, q_{p_k}^{-1} g_{p_k}\rangle$. Then, for every $p_k \in P$, E_{p_k} is a subgroup of index q_{p_k} in H_{p_k} and $r(E_{p_k}) = 1$. For any cardinal $m \le r(G)$, we consider the subgroup $A_m = \langle \bigoplus_{p_k \in P^{(m)}} E_{p_k}, q_{p_1}^{-1} g_{p_1} + q_{p_2}^{-1} g_{p_2}, q_{p_1}^{-1} g_{p_1} + q_{p_3}^{-1} g_{p_3}, \ldots, q_{p_1}^{-1} g_{p_1} + q_{p_m}^{-1} q_{p_m}\rangle$, where

$P^{(m)}$ is a subset of cardinal m of P. Following the same reasoning as in Theorem 1, we obtain that A_m is indecomposable.

Corollary 5. *For every $p \in P$, we consider the group $Q^{(p)}$ of all rational numbers whose denominators are powers of p. Then the group $G = \bigoplus_{p\in P} Q^{(p)}$ has indecomposable subgroups of every rank $m \leq r(G)$.*

Proof. For every $p \in P$, $t(Q^{(p)}) = (0, \dots, 0, \infty, 0, \dots)$, where ∞ stands at the proper place of the p-height h_p. So the group G satisfies the conditions from Corollary 4.

Corollary 6. *If I is a index set with $|I| \leq |P|$, then the group $Q^* = \oplus_I Q$ has indecomposable subgroups of every rank $m \leq |I|$.*

From Corollary 3 it follows:

Corollary 7. *We consider G a reduced, torsion-free of rank one group, I at most countable index set and let be $G^* = \oplus_I G$. If there is a set $P_0 = \{p_1, p_2, \dots, p_n, \dots\}$ of distinct prime numbers with the property that, for every $p_k \in P_0$ there is $g_k \in G$ (not necessarily distinct) such that $h^G_{p_k}(g_k) = \infty$, and there is $q \in P \setminus P_0$, for which there is $\overline{g}_k \in G_k$ such that $h^G_q(\overline{g}_k) = 1$, then G has indecomposable subgroups of every rank $m \leq |P_0|$.*

One can notice that there is a basic condition in all the cases we have mentioned above: the direct summands of group G have elements of infinite p-height, for certain prime numbers p. Afterwards this condition is replaced by another: the existence of a rigid system in group G. For the beginning we generalize [3,88.3].

Theorem 8. *Let be $\{H_i | i \in I\}$ a family of torsion-free groups such that, for every $i \in I$, there is $G_i \leq H_i$, where $\{G_i | i \in I\}$ is a rigid system of groups, with the property that there is a set $P_0 = \{p_i | i \in I\}$ of prime numbers (not necessarily distinct) such that, for every $p_i \in P_0$, there is a $g_i \in G_i$ with $h^{H_i}_{p_i}(g_i) = 1$ and which is not divisible by p_i in G_i. Then the group $H = \bigoplus_{i\in I} H_i$ has indecomposable subgroups of every rank $m \leq |I|$.*

Proof. Let m be any cardinal, $m \leq |I|$ and let $I^{(m)}$ be a subset of cardinal m of I. According to the hypothesis, for every $i \in I$, there is a $p_i \in P_0$ for which there is a $g_i \in G_i$ which is not divisible by p_i in G_i. Now we consider the subgroup $A_m = \langle \bigoplus_{i_j \in I^{(m)}} G_{i_j}, p_{i_1}^{-1} g_{i_1} + p_{i_2}^{-1} g_{i_2}, p_{i_1}^{-1} g_{i_1} + p_{i_3}^{-1} g_{i_3}, \ldots, p_{i_1}^{-1} g_{i_1} + p_{i_m}^{-1} g_{i_m} \rangle$ of H. Then, for every $p_i \in P_0$, g_i is not divisible by p_i in A. Since $\{G_i | i \in I\}$ is a rigid system of groups, for every $i \in I$, G_i is fully invariant in H; so, for every $i \in I$, G_i is fully invariant in A_m. We suppose that $A_m = B_m \oplus C_m$. Then, for every $i_j \in I^{(m)}$, $G_{i_j} = (G_{i_j} \cap B_m) \oplus G_{i_j} \cap C_m)$. Since each G_{i_j} is indecomposable, it follows that, for every $i_j \in I^{(m)}$, either $G_{i_j} \cap B_m = 0$ or $G_{i_j} \cap C_m = 0$. So, for every $i_j \in I^{(m)}$, either $G_{i_j} \subseteq B_m$ or $G_{i_j} \subseteq C_m$. We suppose that there is $j \neq 1$ such that $G_{i_1} \subseteq B_m$ and $G_{i_j} \subseteq C_m$. In this case we consider the element $p_{i_1}^{-1} g_{i_1} + p_{i_j}^{-1} g_{i_j} = b_m + c_m$, with $b_m \in B_m$ and $c_m \in C_m$. It follows that $p_{i_1} | g_{i_1}$ and $p_{i_j} | g_{i_j}$ in A_m, which is impossible, according to the hypothesis. Therefore, for every $i_j \in I^{(m)}$, either $G_{i_j} \subseteq B_m$ in which case $C_m = 0$ or $G_{i_j} \subseteq C_m$ in which case $B_m = 0$, because $\bigoplus_{i_j \in I^{(m)}} G_{i_j}$ is essential in A_m (in this way it is straightforward to verify that $A_m / (\bigoplus_{i_j \in I^{(m)}} G_{i_j})$ is torsion). It follows that A_m is indecomposable and since $r(A_m) = |I^{(m)}|$, the theorem is completely proved.

An immediate consequence of Theorem 8 is:

Corollary 9. *Let $\{H_i | i \in I\}$ be a family of torsion-free groups such that, for every $i \in I$, there is $G_i \leq H_i$, where $\{G_i | i \in I\}$ is a family of reduced of rank one groups, with the property that, for every $i_1, i_2 \in I$, $i_1 \neq i_2$, $t(G_{i_1})$ and $t(G_{i_2})$ are incomparable. Then the group $H = \bigoplus_{i \in I} H_i$ has indecomposable subgroups of every rank $m \leq |I|$.*

Proof. According to the hypothesis, $\{G_i | i \in I\}$ is a rigid system of groups and there is a set $P_0 = \{p_i | i \in I\}$ of prime numbers (not necessarily distinct) such that, for every $p_i \in P_0$, there is a $g_i \in G_i$ with $h_{p_i}^{H_i}(g_i) = 1$ and which is not divisible by p_i in G_i. Since $|I| = r(G)$, the statement follows from Theorem 8.

Let $G = B \oplus C$ be any group and A a subgroup of G. According to [2,p.44], there are subgroups B_2, B_1 of B and there are subgroups C_2, C_1 of C such that $B_2 \leq B_1$, $C_2 \leq C_1$, $B_1 \oplus C_1$ is the minimal direct sum containing A, and $B_2 \oplus C_2$ is the maximal direct sum contained in A, with components in B, respectively C. So $B_2 \oplus C_2 \leq A \leq B_1 \oplus C_1$ and it is straightforward to verify that A is a subdirect sum of B_1 and C_1 with kernels B_2, respectively C_2. Then, according to [2,p.43,44] the following relationships hold:

$$A/(B_2 \oplus C_2) \cong B_1/B_2 \cong C_1/C_2 \tag{3}$$

$$(B_1 \oplus C_1)/A \cong A/(B_2 \oplus C_2) \tag{4}$$

$$A/B_2 \cong C_1 \tag{5}$$

$$A/C_3 \cong B_1. \tag{6}$$

Remark 10. *Let $G = B \oplus C$ be any group and A a subgroup of G. If C is free and A is indecomposable, then either $r(A) = I$ or $A \subseteq B$.*

Proof. According to the hypothesis, keeping the above notation, C_1 is free. In this case, the relationship (4) and [2,14.4] show that B_2 is a direct summand in A - which is in contradiction to the hypothesis.

Now, we suppose that $G = B \oplus C$ is a torsion-free group, with $r(B) = r(C) = 1$. Then, according to the relationships (3), $B_2 = 0$ if and only if $C_2 = 0$; in this case, according to the condition (4), $(B_1 \oplus C_1)/A \cong A$ - which is impossible, because $(B_1 \oplus C_1)/A$ is torsion and A is torsion-free. Therefore $B_2 \neq 0$ and $C_2 \neq 0$. On the other hand, if $B_1 = B_2$ then $C_1 = C_2$ (see (3)) and in this case $A = B_2 \oplus C_2$.

Of course $B_2 \oplus C_2$ is essential in A ($A/(B_2 \oplus C_2)$ is torsion), $B_2 = B \cap A$, and $C_2 = C \cap A$. It follows that if B_2 is a proper subgroup of B_1, then also C_2 is a proper subgroup of C_1 and

$$A = \langle B_2 \oplus C_2, a_1, a_2, \ldots \rangle \tag{7}$$

where, for every $i = 1, 2, \ldots$, there is a $b_1^i \in B_1 \setminus \{0\}$ and there is a $c_1^i \in C_1 \setminus \{0\}$ such that $a_i = b_1^i + c_1^i$.

Now we can present the structure of indecomposable subgroups of completely decomposable groups of rank 2.

Theorem 11. *Let $G = B \oplus C$ be a torsion-free group with $r(B) = r(C) = 1$ and let A be any subgroup, of the form (7), of G. Then the following statements are equivalent:*

a) A is indecomposable;

b) i) for every $a_i \in A \setminus (B_2 \oplus C_2)$, there are $b_2^i \in B_2 \setminus \{0\}$ and $c_2^i \in C_2 \setminus \{0\}$ for which there are the prime numbers p_2^i and q_2^i (not necessarily distinct) such that b_2^i is not divisible by p_2^i in B_2, c_2^i is not divisible by q_2^i in C_2 and $a_i = (p_2^i)^{-1} b_2^i + (q_2^i)^{-1} c_2^i$;

ii) the subgroups B_2 and C_2 are fully invariant in A.

Proof. In view of Theorem 8, suffice it to show that a) implies b). Let $A = \langle B_2 \oplus C_2, b_1 + c_1, b_2 + c_2, \ldots \rangle$ be a subgroup, of the form (7), of G, where $b_1, b_2, \dots \in B \setminus B_2$ and $c_1, c_2, \dots \in C \setminus C_2$. According to the hypothesis, B_2 and C_2 are reduced and not pure in B and C respectively. Let p be a prime number and let $b + c + (B_2 \oplus C_2)$ be an element of order p from $A/(B_2 \oplus C_2)$. Then $pb = x \in B_2$ and $pc = y \in C_2$. If x is divisible by p in B_2, then $b \in B_2$, what is in contradiction to the hypothesis. It follows that x is not divisible by p in B_2 and $b = p^{-1}x$. Analogously it follows that $c = p^{-1}y$ and the statement i) from point b) is completely proved.

If $b + c + (B_2 \oplus C_2)$ is an element of order p^r, with $r \geq 2$, then we follow the same reasoning.

For the proof of the second statement from point b) we distinguish two cases:

Case 1. $t(B_2)$ and $t(C_2)$ are incomparable. Then this gives the required result.

Case 2. $t(B_2) \leq t(C_2)$. In this case there is a monomorphism $f : B_2 \to C_2$; so $B_2 \cong f(B_2) = B_2^* \leq C_2$. We consider the group $A^* = \langle B_2^* \oplus C_2, a_1^*, a_2^*, \ldots \rangle$, where, for every $i = 1, 2, \ldots$ $a_i^* = (p_2^i)^{-1}(b_2^i)^* + (q_2^i)^{-1}c_2^i$, and $(b_2^i)^* = f(b_2^i) \in B_2^*$; also we consider the subgroup $C_3 = \langle C_2, a_1^*, a_2^*, \ldots \rangle$ of A^*. Then, for every $i = 1, 2, \ldots$ there is $n_i \in N^*$ such that $n_i a_i^* \in C_2$. We are going to show that $A^* = B_2^* \oplus C_3$. Of course $A^* = B_2^* + C_3$. Let a^* be any element from A^*. We suppose that there are $x^*, y^* \in B_2$ and there are $u, v \in C_2$, such that $a^* = x^* + u + a_i^* = y^* + v + a_j^*$. Let be $n \in N^*$ such that $n(a_j^* - a_i^*) \in C_2$. Then $n(x^* - y^*) = n(v - u) + n(a_j^* - a_i^*) = 0$. Since G is torsion-free, it follows that $x^* = y^*$ and $u + a_i^* = v + a_j^*$, that is a^* may be written in a unique way of the form $b^* + c$, with $b^* \in B_2^*$ and $c \in C_3$. Since $A \cong A^*$, it follows that A is completely decomposable, what is in contradiction to the hypothesis.

From Remark 10 or Theorem 11 we obtain:

Corollary 12. *If B is a torsion-free of rank one group, then the group $G = B \oplus Z$ has no indecomposable subgroups of rank 2.*

Proof. If G is a group as in the statement, then there is no direct sum in G which is not made up of fully invariant direct summands. Now the statement follows from Theorem 11.

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